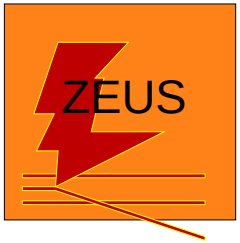
On behalf of H1 and ZEUS Collaborations



Exclusive Diffraction at HERA

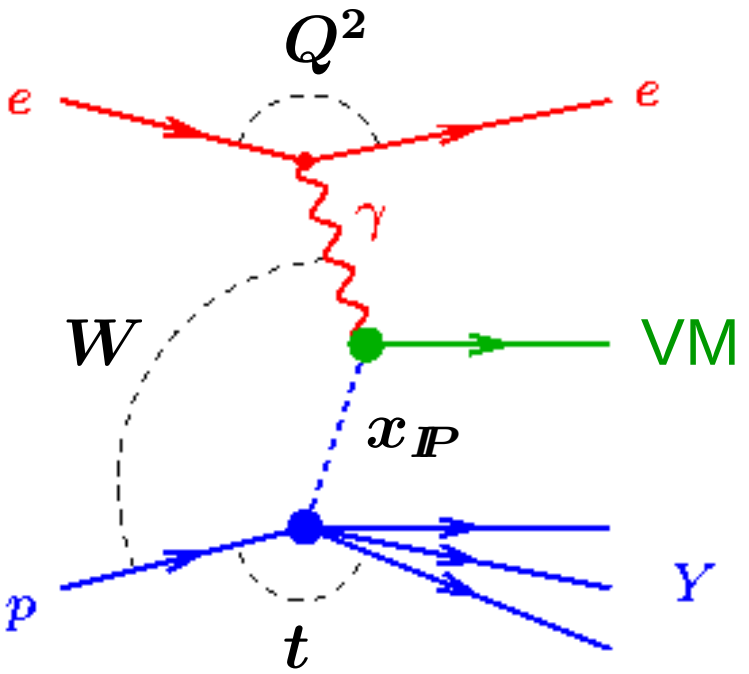
Xavier Janssen

XXXIX International Symposium on Multiparticle Dynamics

4-9 September 2009, “Golden Sands" ,Gomel Region, Belarus

Diffractive Vector Meson Production and DVCS

$$e + p \rightarrow e + VM\,(= \rho, \phi, J/\psi, ..., or\ \gamma) + Y\,(or\ p)$$



Q^2	Photon Virtuality Photoproduction: $Q^2 \sim 0$
W	γp CMS energy
t	4-momentum transfer squared
$x_{I\!P}$	Momentum fraction of the colour singlet exchange

Regge Theory

= Soft $I\!P$omeron exchange

$$\sigma \propto (\frac{W}{W_0})^{4(\alpha_{I\!P}(t)-1)}$$

$$\alpha_{I\!P}(t) = 1.08 + 0.25\,t \text{ (DL)}$$

Light VM at low Q^2 and low $|t|$

pQCD Models

Exchange of ≥ 2 gluons

$$\sigma \propto (xG(x, Q^2))^2$$

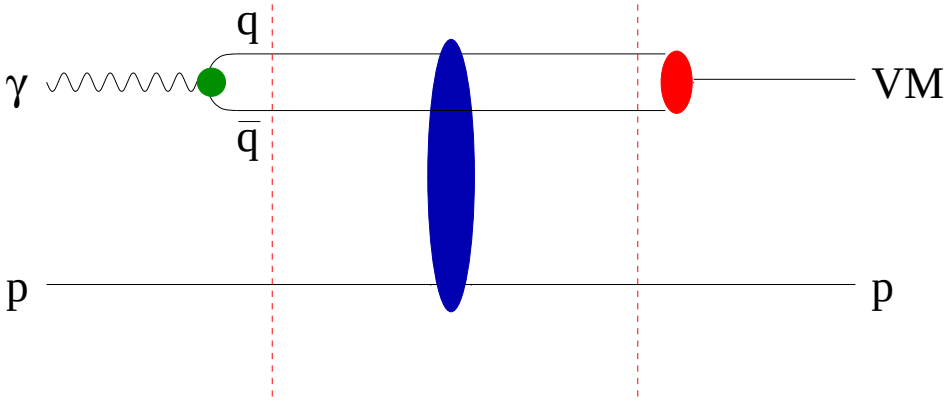
Steep rise of $xG(x, Q^2)$

Requires hard scale: Q^2, t or m_q

$\Longrightarrow$ Investigate transition between soft and hard regimes

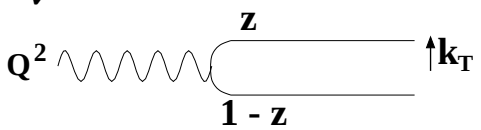
VM theory: Perturbative QCD approaches

Dipole approach (k_t factorisation)



$$\mathcal{A} = \Psi^{\gamma}_{q\bar{q}} \otimes \sigma_{q\bar{q}-p} \otimes \Psi^{V}_{q\bar{q}}$$

Scanning radius decrease with increasing Q^2 or M_V^2 $\rightarrow \mu^2 = z(1-z)(Q^2 + M_V^2)$



$\rightarrow \sigma_L \propto \frac{Q^2/M_V^2}{(Q^2+M_V^2)^4} [\alpha_s(\mu^2)\, G(x,\mu^2)]^2$

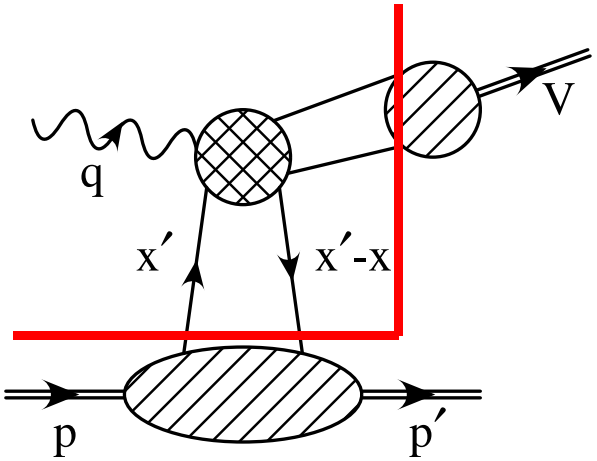
with $z \simeq 1/2 \rightarrow \mu^2 \simeq 1/4(Q^2 + M_V^2)$

$\rightarrow \sigma_T \propto \frac{1}{(Q^2+M_V^2)^4} [\alpha_s(\mu^2)\, G(x,\mu^2)]^2$

with $z = 0, 1$ endpoints contributions

→ hard scale damped

Collinear factorisation theorem



$$\mathcal{A}_L = f(x, x', t, \mu) \otimes H \otimes \Psi^{V}$$

where f_i: non-forward PDF ($x' \neq x$)

→ Generalized Parton Density

Theorem proven for σ_L; often assumed for σ_T

Collins, Frankfurt & Strikman [hep-ph/9611433]

Dipole - Saturation:
- Kowalski, Motyka, Watt (KMW) [hep-ph/0606272]
- Marquet, Peschanski, Soyez (MPS) [hep-ph/0702171]

Dipole - k_T factorisation:
- Ivanov, Nikolaev, Savin (INS) [hep-ph/0501034]

Collinear - GPD:
- Goloskokov, Kroll (GK) [hep-ph/07083569]

Parton hadron duality:
- Martin, Ryskin, Teubner (MRT) [hep-ph/9609448]

VM theory: Main features / expectations

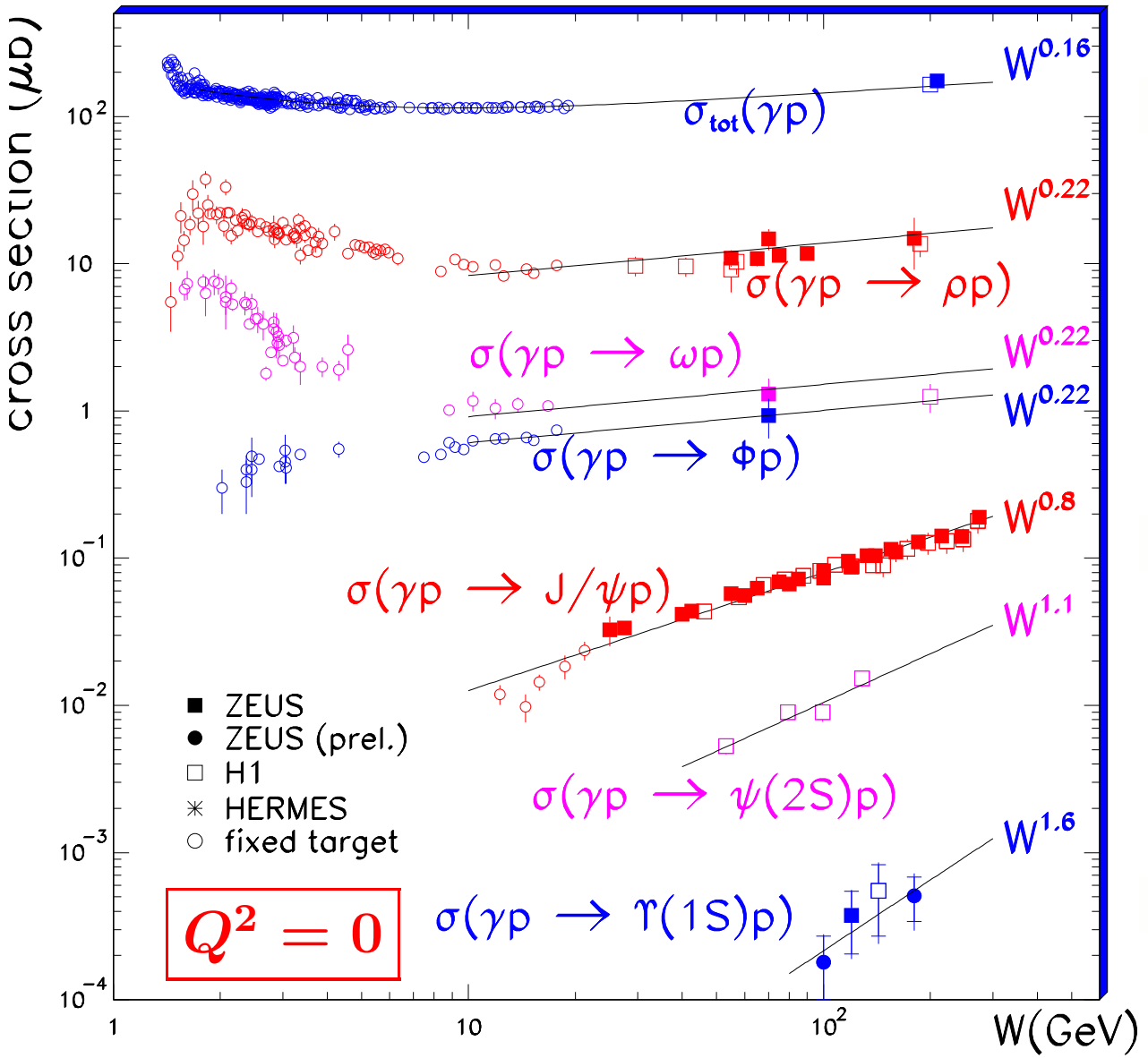
$\sigma(Q^2)$: $\sigma_L \propto Q^{-6}$; $\sigma_T \propto Q^{-8}$ but modified by gluon pdf Q^2 depend., quark Fermi motion and virtuality, $\alpha_s(Q^2)$, higher order.
$\longrightarrow$ Naive $R = \sigma_L/\sigma_T \propto Q^2/M_V^2$ also modified.

$\sigma(W)$:
- For σ_L at high Q^2 and heavy VM, hard (universal) W depend. expected from $1/x$ hard gluon pdf evolution.
- For light VM, delayed approach to hard pQCD regime (σ_T).

dσ/dt: $\propto \exp(-b|t|)$ for low $|t|$, where $b = b_{q\bar{q}} \otimes b_{I\!P} \otimes b_p$
- Expect common b for σ_L at high Q^2 and heavy VM.
  $\longrightarrow$ Naive universality of b vs. $\mu^2 = 1/4(Q^2 + M_V^2)$
- Larger dipole in σ_T than in σ_L $\longrightarrow$ expect $b_T > b_L$
  $\longrightarrow$ Delayed universality of b vs. μ^2

Helicity amplitudes: see later

Soft to hard transition: mass



- Low mass (ρ, ϕ, ω; $M_V^2 \simeq 1$ GeV^2): no pert. scale → weak energy dep. (soft regime)
- High mass (J/ψ, υ): pert. scale → strong energy dep. (hard regime)
- Large mass (Υ) important skewing effect

Upsilon Photoproduction

New ZEUS result:

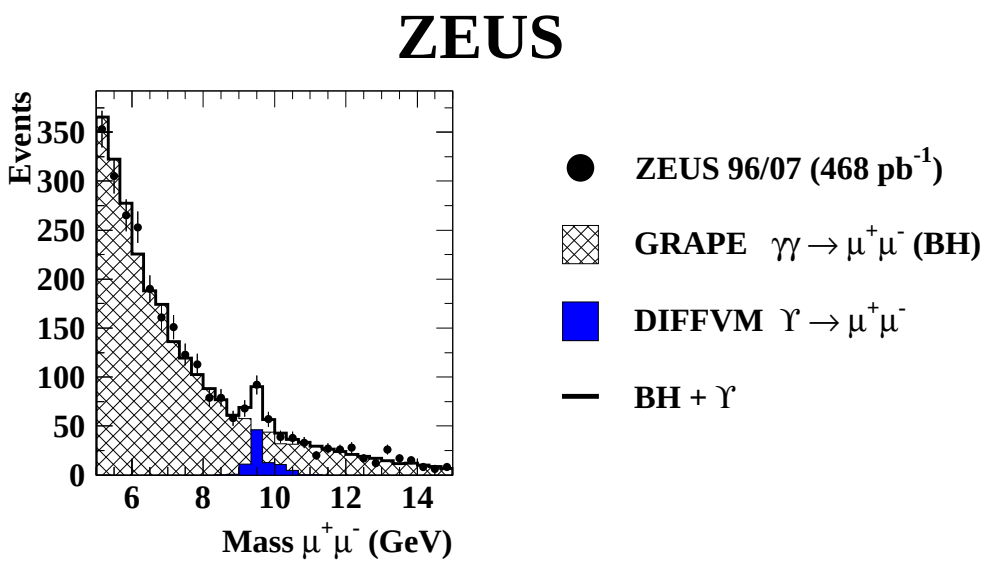
1996-2007 data: 468 pb^{-1}

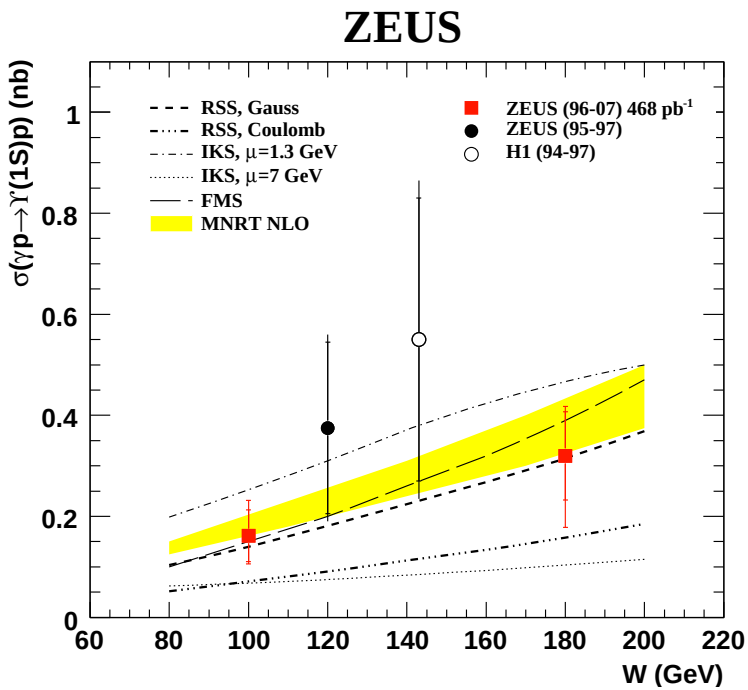
$60 < W < 220$ GeV, $Q^2 < 1$ GeV^2

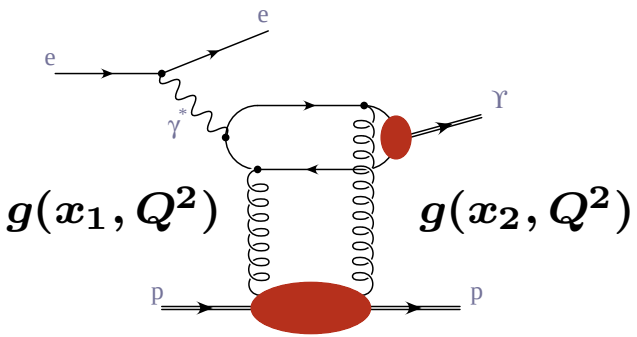
DESY-09-036 (accepted by PLB)

$$e + p \rightarrow e + \Upsilon + p$$

$$\Upsilon \rightarrow \mu^+ + \mu^-$$





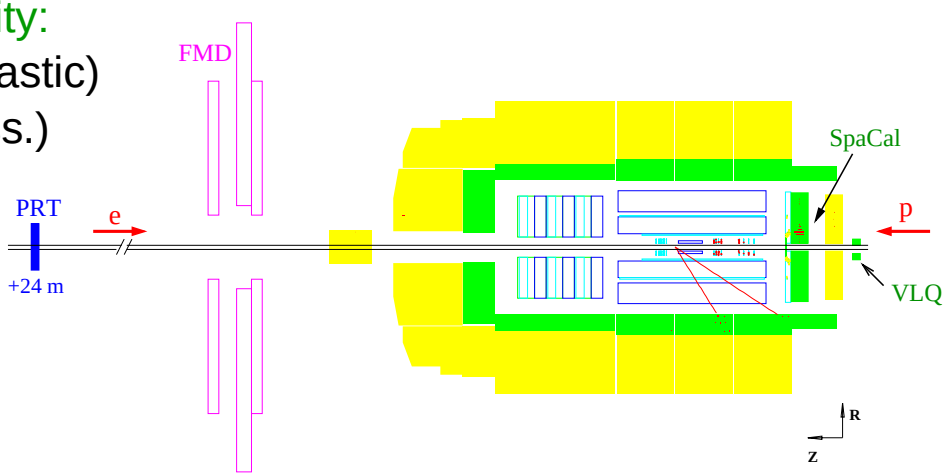


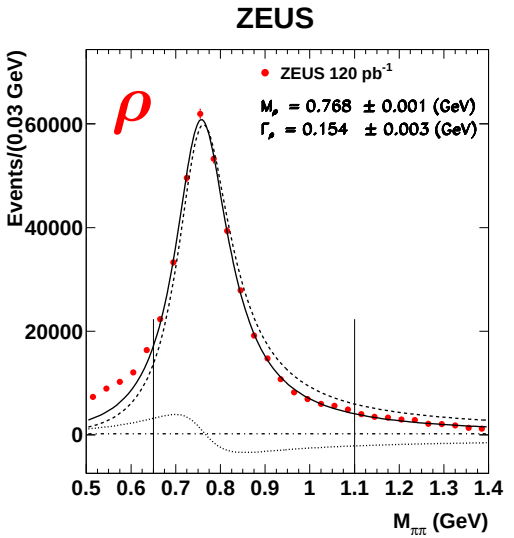
W dependence of the cross section is in agreement with pQCD models including skewing, i.e. $x_1 \neq x_2$

Light VM in DIS: Data Selection

$$e^+ + p \to e^+ + \rho + p \text{ (or } Y) \quad ; \quad \rho \to \pi^+ + \pi^-$$
$$e^+ + p \to e^+ + \phi + p \text{ (or } Y) \quad ; \quad \phi \to K^+ + K^- \quad (\text{BR} = 49\%)$$

Forward activity:
NOTAG ($\simeq$ elastic)
TAG ($\simeq$ p diss.)





H1: ρ and ϕ

elastic and p diss. channels
1996-2000 data: 51 pb^{-1}

$\mathbf{2.5 < Q^2 < 60\ GeV^2}$
$\mathbf{35 < W < 180\ GeV}$
elastic: $\mathbf{|t| < 0.5\ GeV^2}$
p diss.: $\mathbf{|t| < 3\ GeV^2}$

H1prelim-08-013 & H1prelim-09-017

ZEUS: ρ

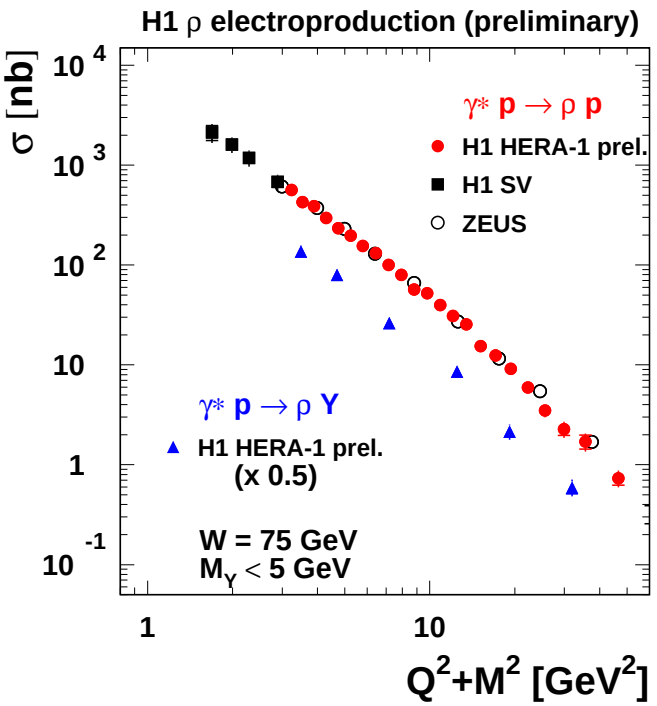
mostly elastic channel
1996-2000 data: 119 pb^{-1}

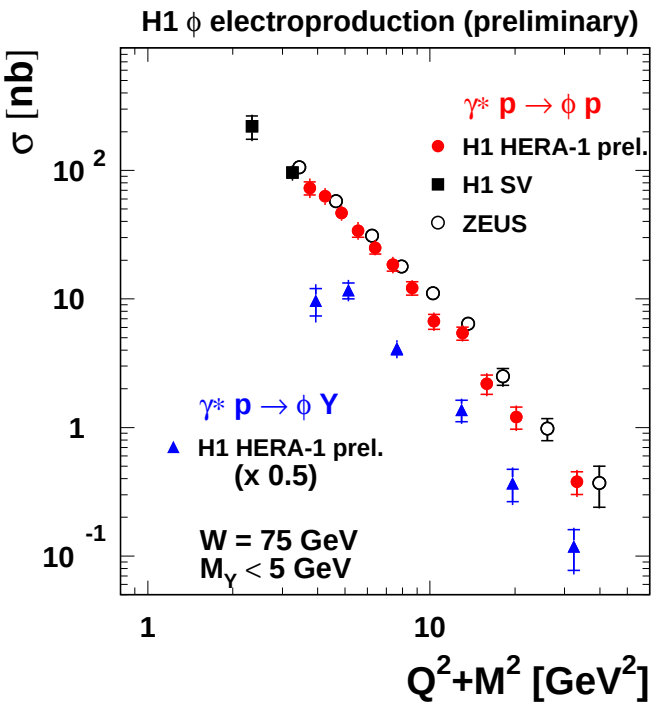
$\mathbf{2 < Q^2 < 160\ GeV^2}$
$\mathbf{32 < W < 180\ GeV}$

$\mathbf{|t| < 1\ GeV^2}$

DESY-07-118 (PMC Physics A 1,6)

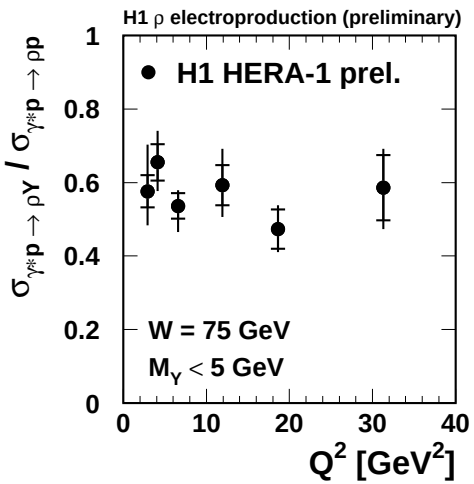
Light VM Cross-sections : Q^2 dependence

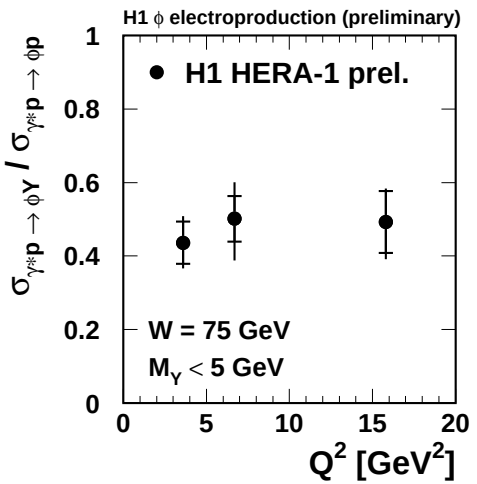


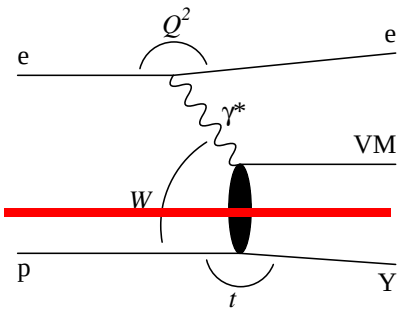


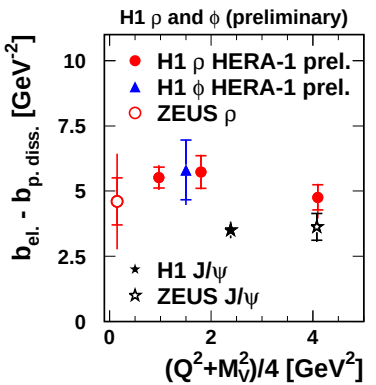
- High precision for elastic cross-sections
- First ϕ p-diss. cross-section
- H1 Zeus relative agreement

Test of vertex ("Regge") factorisation:







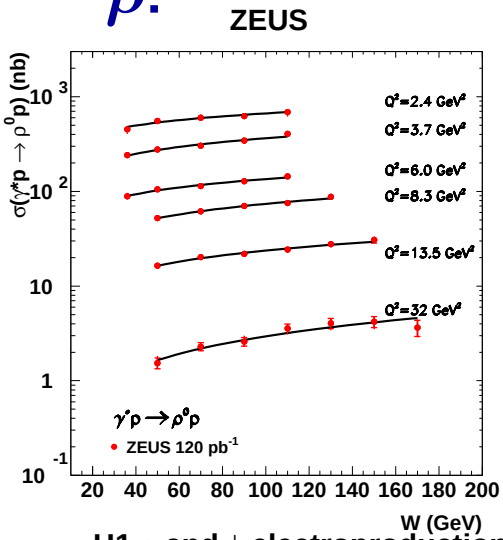


- p.diss/el : no Q^2 dep.
- t-depend. : no Q^2 dep.

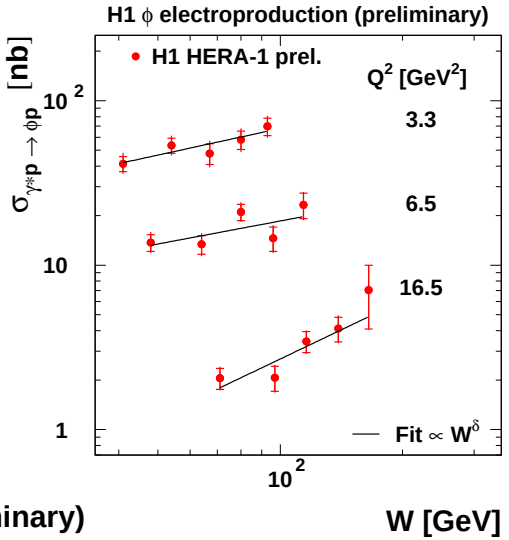
  → vertex factorisation

Soft to hard transition: Q^2

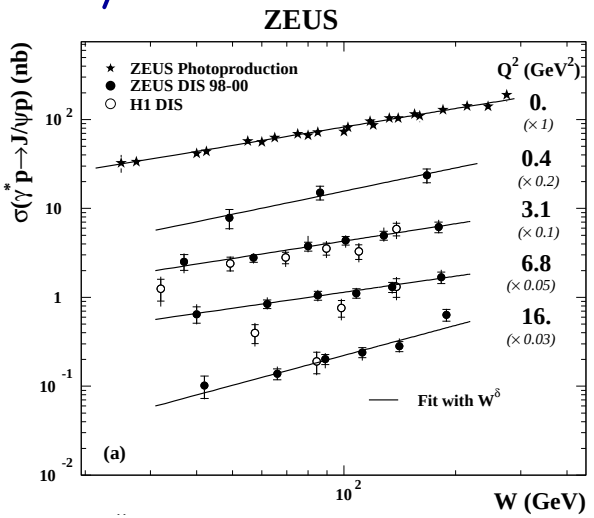
ρ:

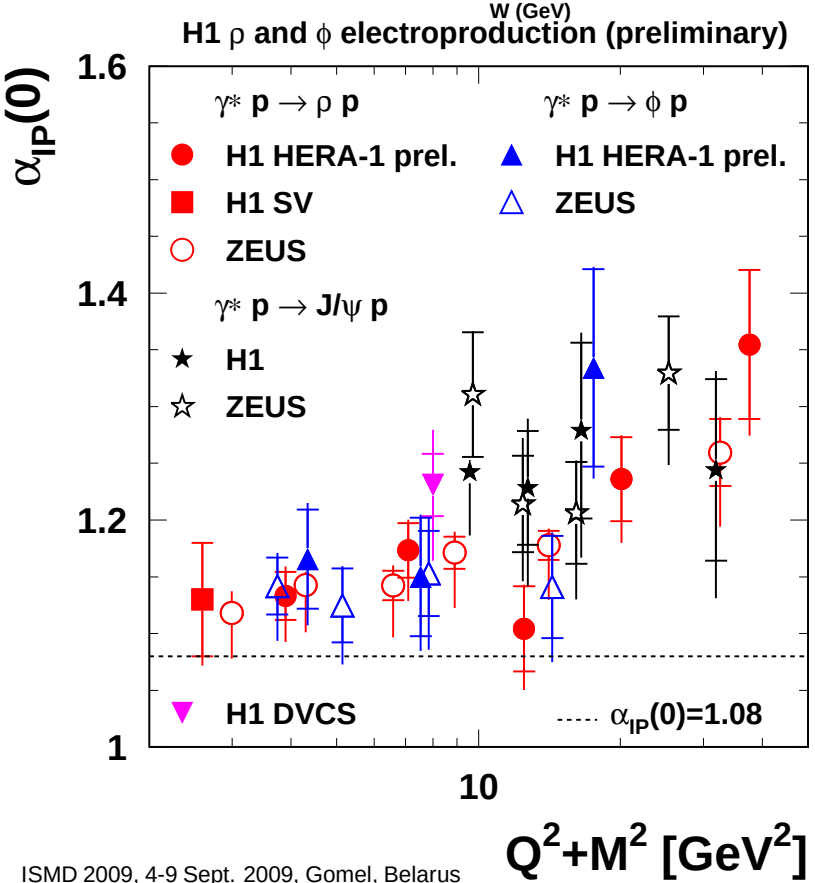


Φ:



J/Ψ:





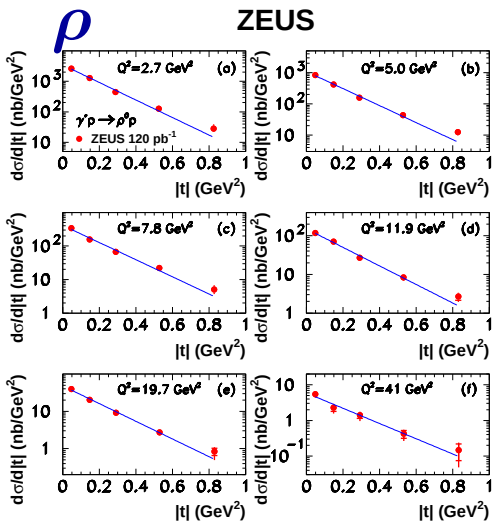
$$\alpha_{I\!P}(0) = 1 + \delta/4 + \alpha'_{I\!P}/\langle|t|\rangle$$

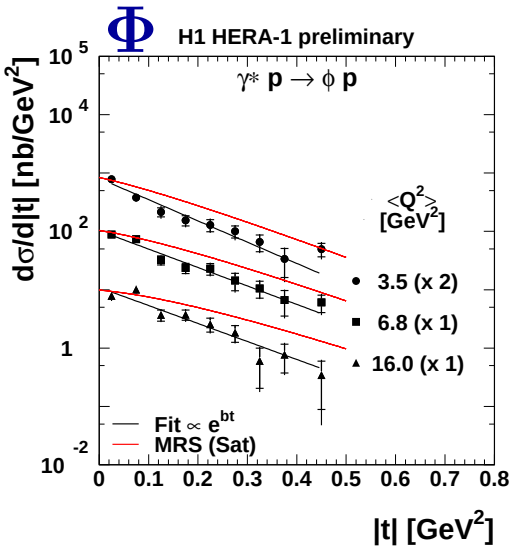
$$\alpha'_{I\!P} = 0 - 0.25\ \mathrm{GeV}^{-2}$$

- Common hardening of $\alpha_{I\!P}(0)$ with $Q^2 + M^2$ for all VM and DVCS

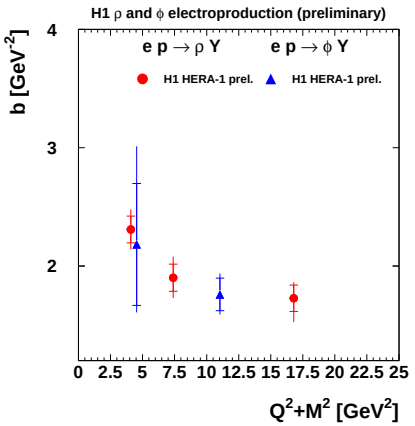
  $\Rightarrow$ Transition from soft to hard regime with $Q^2 + M^2$
- Soft contributions (in σ_L ?) up to $Q^2 \sim 20$ GeV^2 for ρ and ϕ

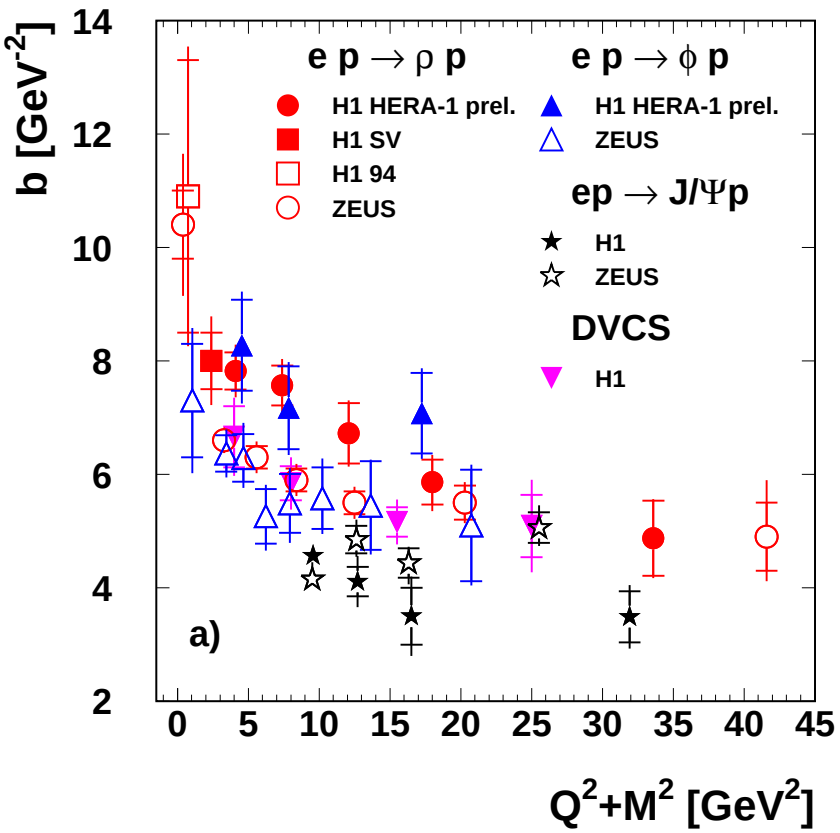
t dependences: Universality and hard diffraction





Proton dissociation:





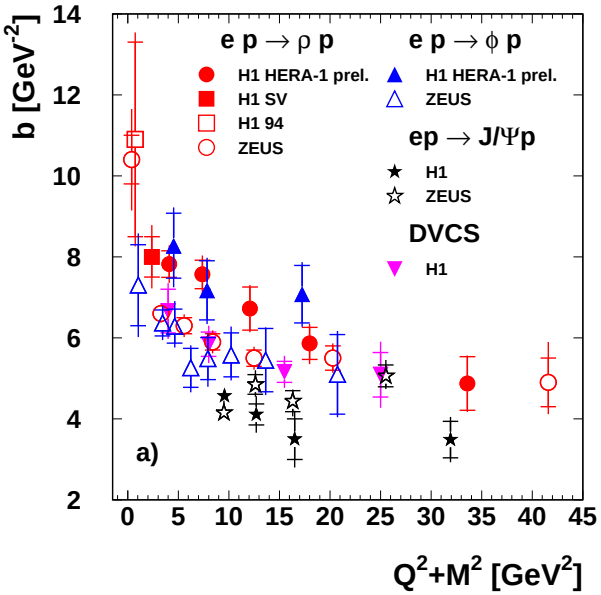
fit $e^{-b|t|}$: $b = b_p \otimes b_{q\bar{q}} \otimes b_{I\!P}$

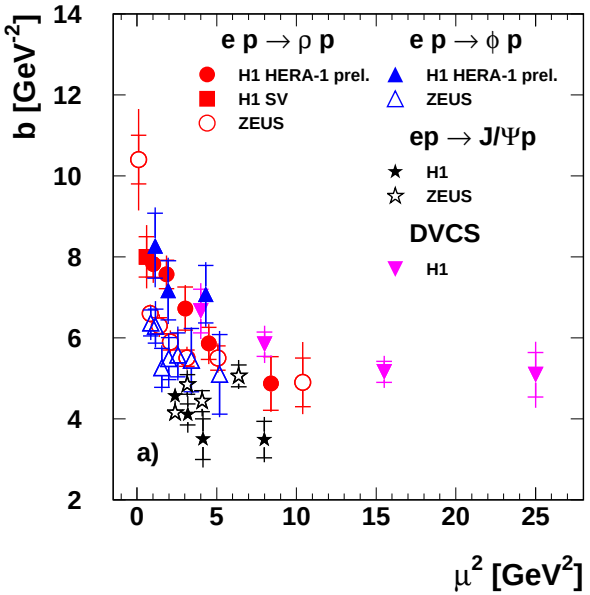
$\rightarrow b \propto q\bar{q}$ dipole size

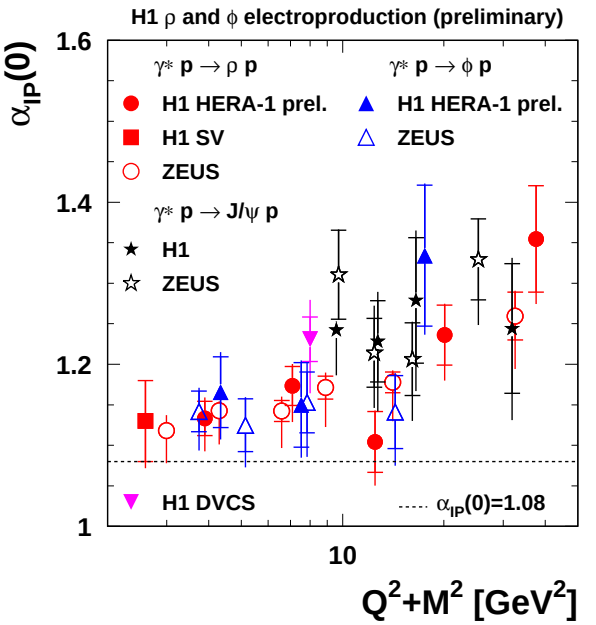
- b_ρ and b_ϕ decrease $Q^2 + M^2$
- Common value with J/ψ for $Q^2 + M^2 > 20$ GeV^2
- Large dipole for light VM at low Q^2

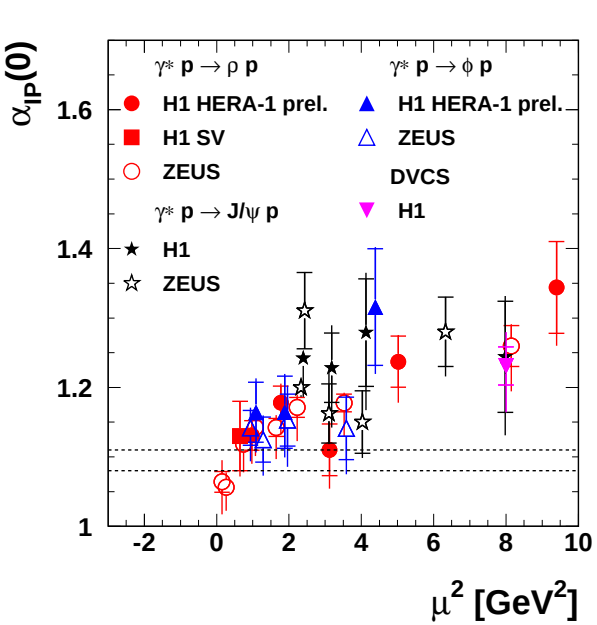
$\Rightarrow$ Transition from soft to hard regime with $Q^2 + M^2$

Note on the scale and universality

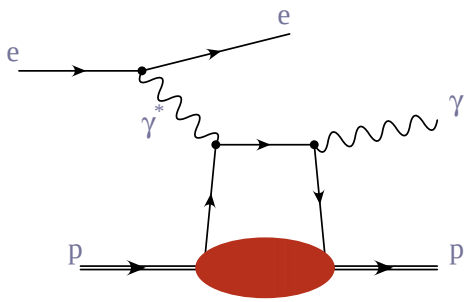








DVCS is like DIS (at LO):



Photon interacts directly with a resolved quark

→ Hard scales are:

for DVCS: $\mu^2 = Q^2$

for VM: $\mu^2 = \frac{Q^2+M^2}{4}$

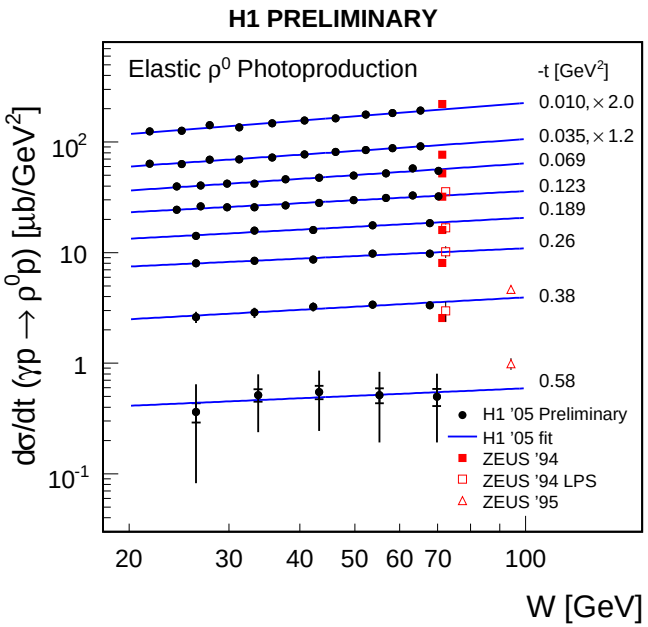
→ Universalality vs μ^2:

Soft/hard transition around $\mu^2 \sim 5\ \mathrm{GeV}^2$

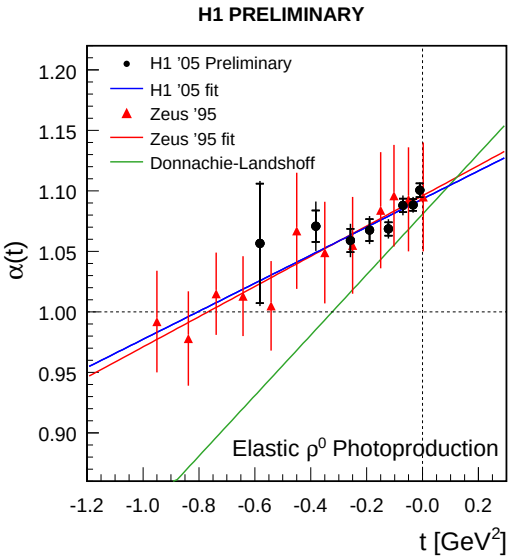
Shrinkage : $\alpha'_{I\!P}$ measurements

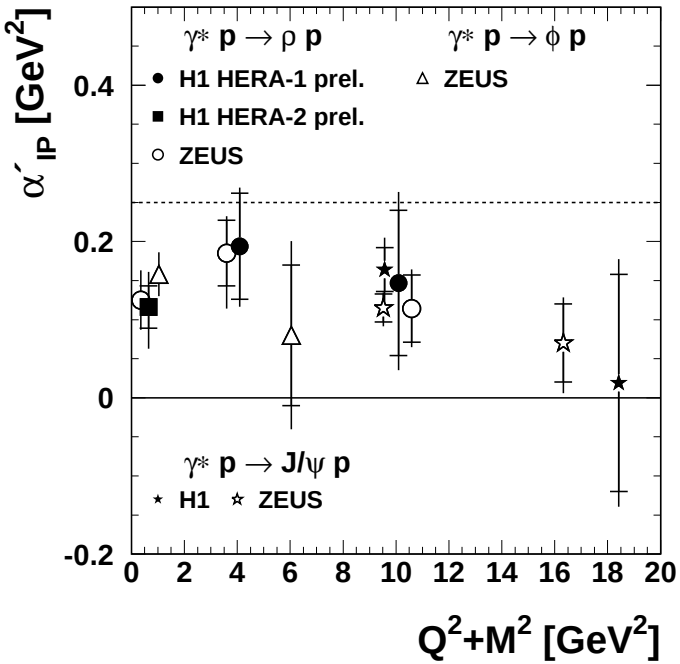
H1 ρ photoproduction measurements:

$$\frac{d\sigma}{dt}(W) \propto e^{b_0 t} W^{4(\alpha_{I\!P}(t)-1)}$$



1. Study W depend. in bins of t:
   $\rightarrow$ Fit: $W^{\delta} \rightarrow \alpha_{I\!P}(t) = 1 + \delta/4$
2. Study $\alpha_{I\!P}(t)$ trajectories:
   $\rightarrow$ Fit: $\alpha_{I\!P}(t) = \alpha_{I\!P}(0) + \alpha'_{I\!P} t$



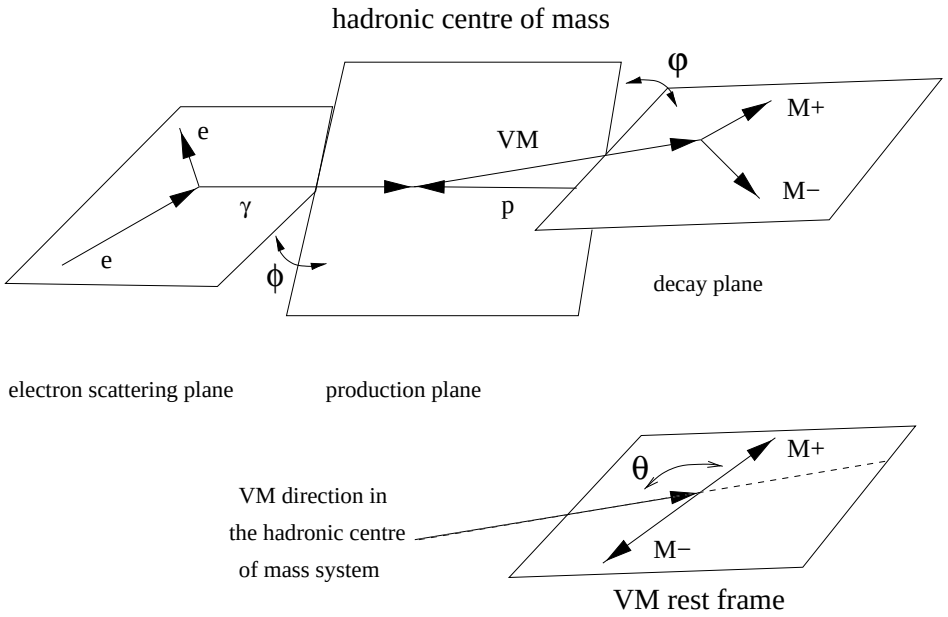


$\Rightarrow$ For all VM, $\alpha'_{I\!P}$ smaller than 0.25 (DL, $p\bar{p}$)
(cf BFKL, multiple $I\!P$ exchange)

SPIN DENSITY MATRIX ELEMENTS

θ^* , Φ , $\varphi \Longleftrightarrow$ 15 SDMEs : $r^{ij}_{kl} \propto T_{\lambda'_\rho \lambda'_\gamma} T_{\lambda_\rho \lambda_\gamma}$

$T_{\lambda_\rho \lambda_\gamma}$: helicity amplitudes



No helicity flip: $T_{00} : \gamma_L \longrightarrow \rho_L$

$T_{11} : \gamma_T \longrightarrow \rho_T$

Single flip: $T_{01} : \gamma_T \longrightarrow \rho_L$

$T_{10} : \gamma_L \longrightarrow \rho_T$

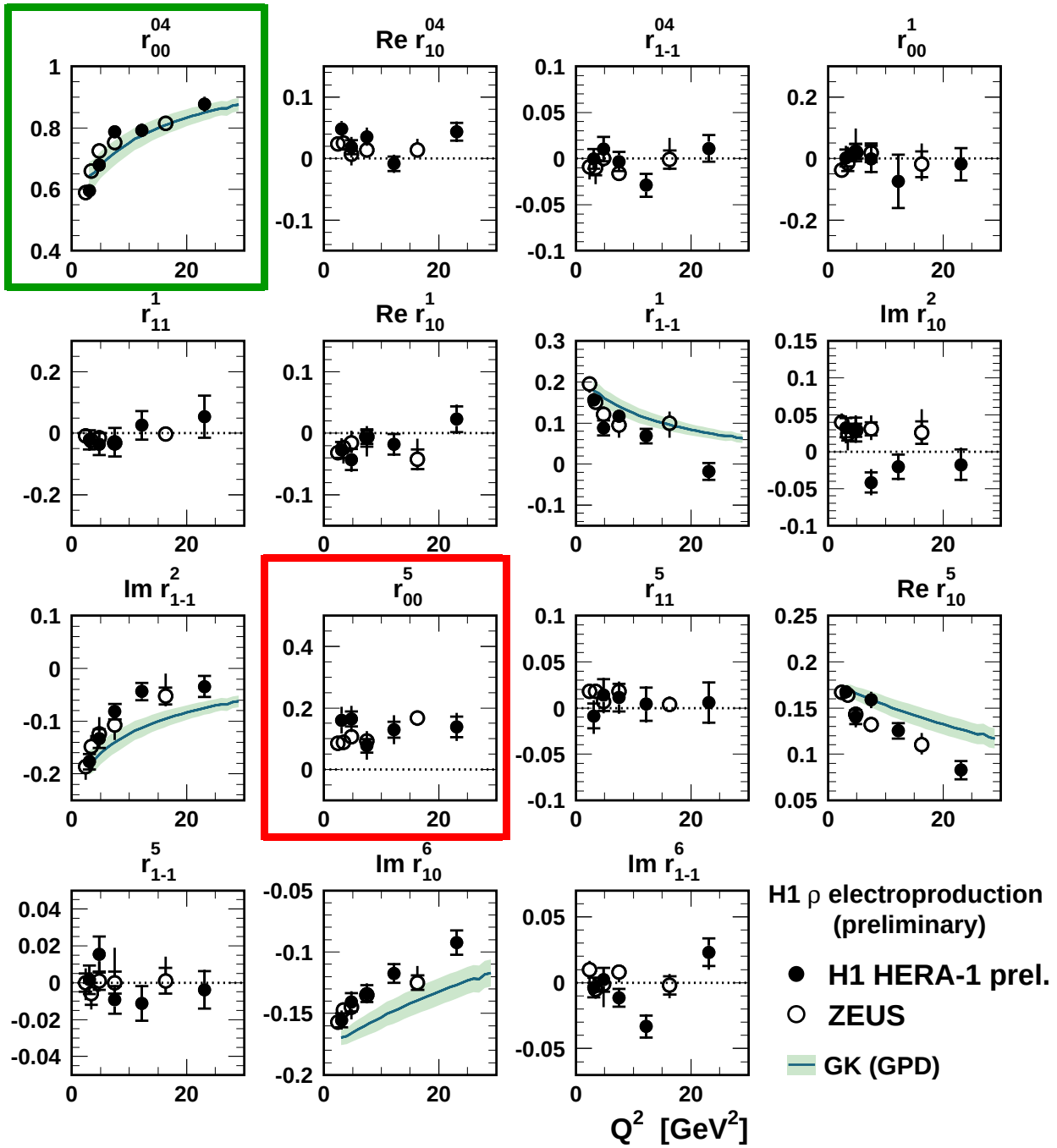
Double flip: $T_{1-1} : \gamma_T \longrightarrow \rho_T$

s-Channel Helicity Conservation (SCHC): $T_{01} = T_{10} = T_{1-1} = 0$

pQCD models:
D. Yu Ivanov and R. Kirschner [hep-ph/9807324]

- SCHC violation (single flip $\propto \sqrt{|t|}$, double $\propto |t|$)
- Hierarchy: $|T_{00}| > |T_{11}| > |T_{01}| > |T_{10}| > |T_{1-1}|$

ρ Polarisation - SDMEs vs. Q^2



- r_{00}^{04} increases with Q^2

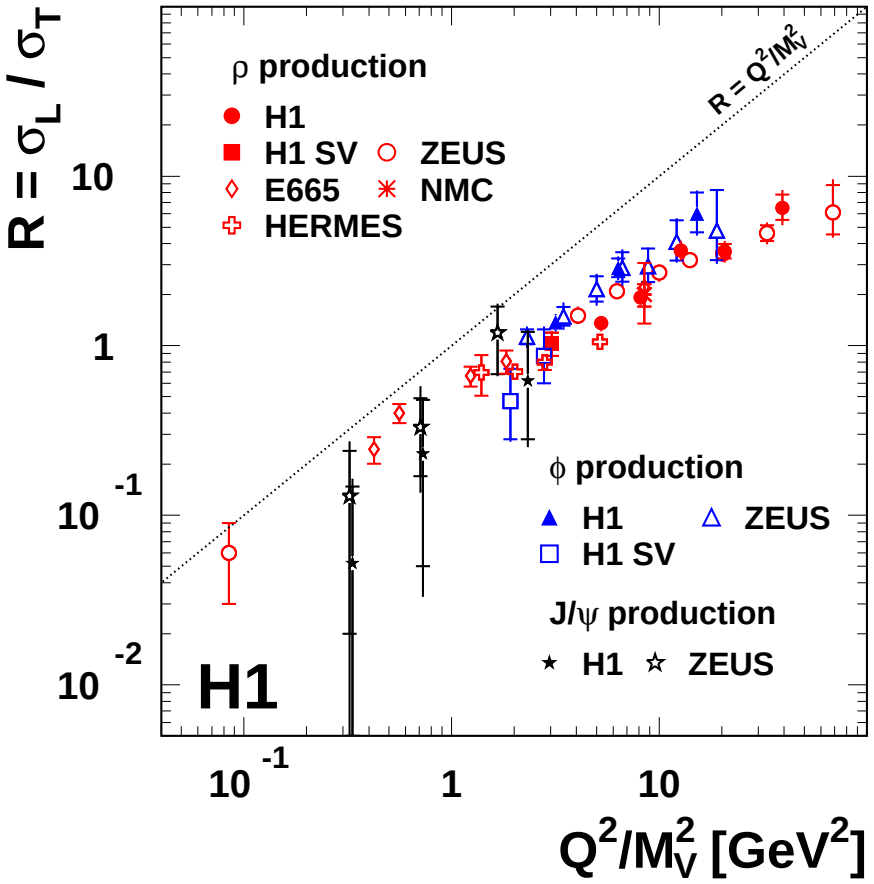
$\leftrightarrow$ similar effects for r_{1-1}^{1}, Im r_{1-1}^{2}, Re r_{10}^{5} and Im r_{10}^{6} (in SCHC)

$\leftrightarrow$ Fair description by GK (GPD) model

- r_{00}^{5} violates SCHC
- Other SDME $\simeq 0$

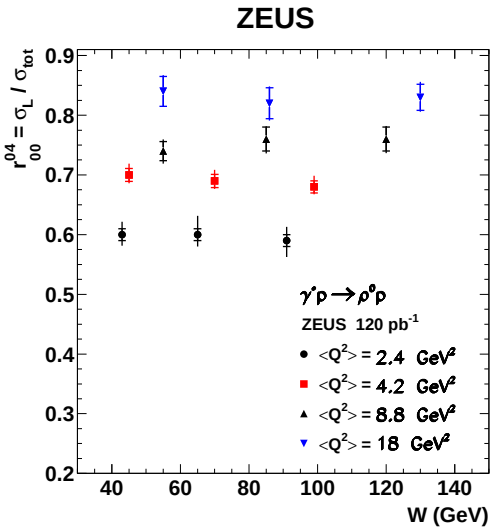
ρ and ϕ Polarisation - $R = \sigma_L / \sigma_T$

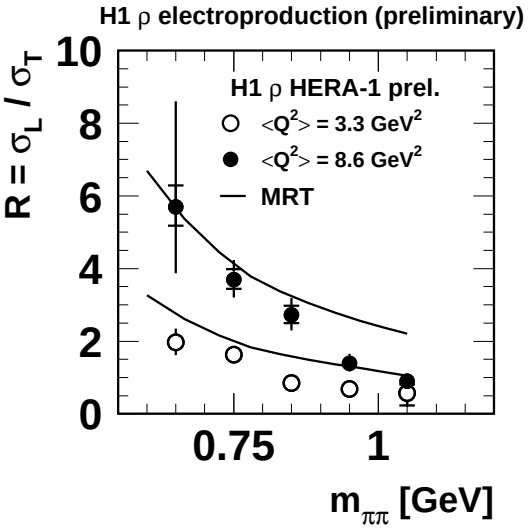
$$R_{SCHC} = \frac{1}{\epsilon}\frac{r_{00}^{04}}{1-\epsilon r_{00}^{04}} = \frac{|T_{00}|^2}{|T_{11}|^2}$$ + non SCHC corrections in H1 case



- Formal pQCD: $R \propto Q^2/M^2$
- Scaling for all VM with Q^2/M_V^2
- Damping at large Q^2

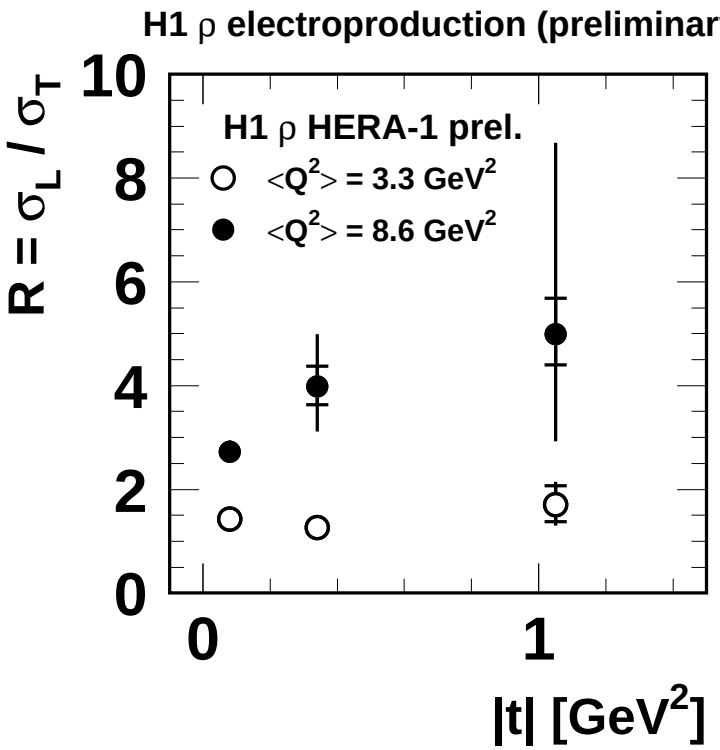
- ρ: no W dependance of R

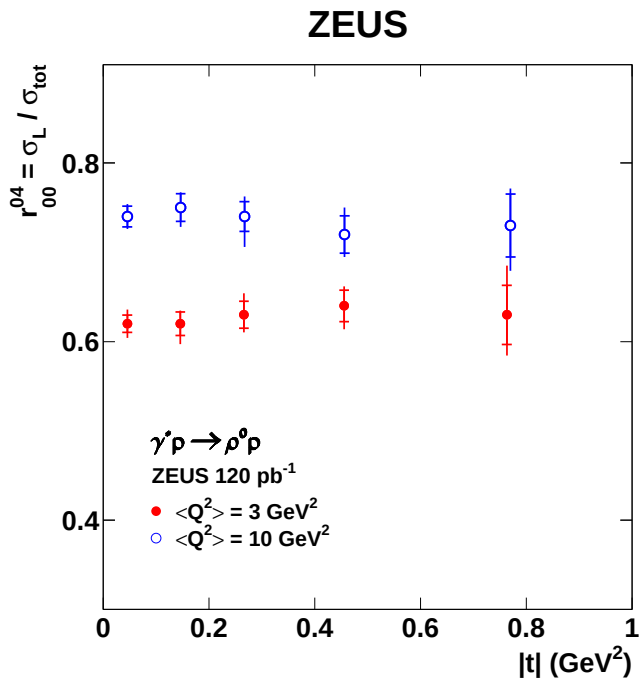




- Strong invariant mass dependance in ρ case
- formal pQCD: $R \propto Q^2/M^2$ but M being diquark mass
  cf Martin, Ryskin, Teubner calculation

ρ and ϕ Polarisation - $R(t)$ and $b_L - b_T$





$$R(t) \propto \frac{\sigma_L}{\sigma_T} \exp(-(b_L - b_T)|t|)$$

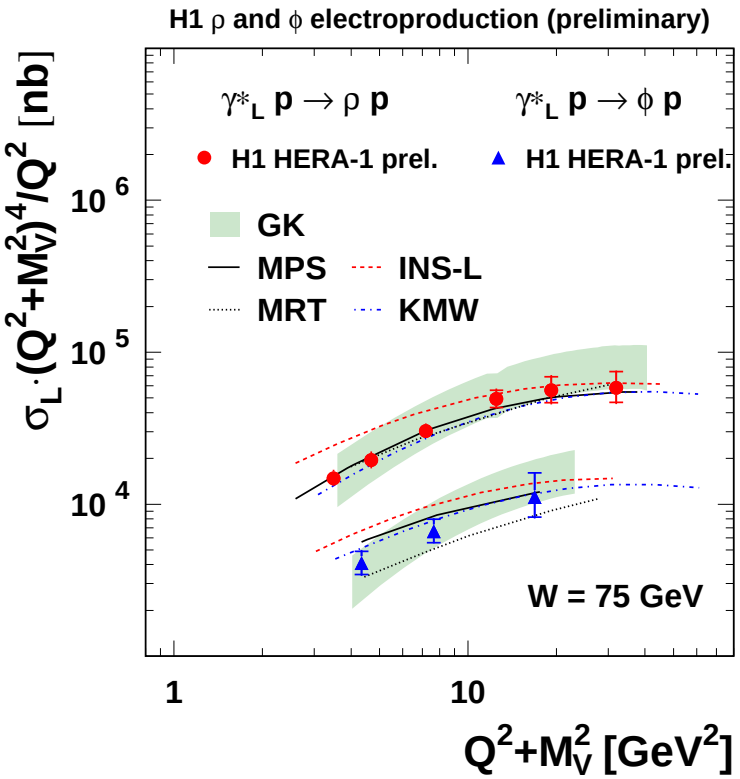
- H1: $(b_L - b_T) < 0$ by 1.5σ for $Q^2 > 5\mathrm{GeV}^2$
- also a t dependance of T_{11}/T_{00} - see later

→ Small difference of transverse size of $q\bar{q}$ dipoles from transverse and longitudinal photons

ρ and ϕ Polarisation - Cross-sections

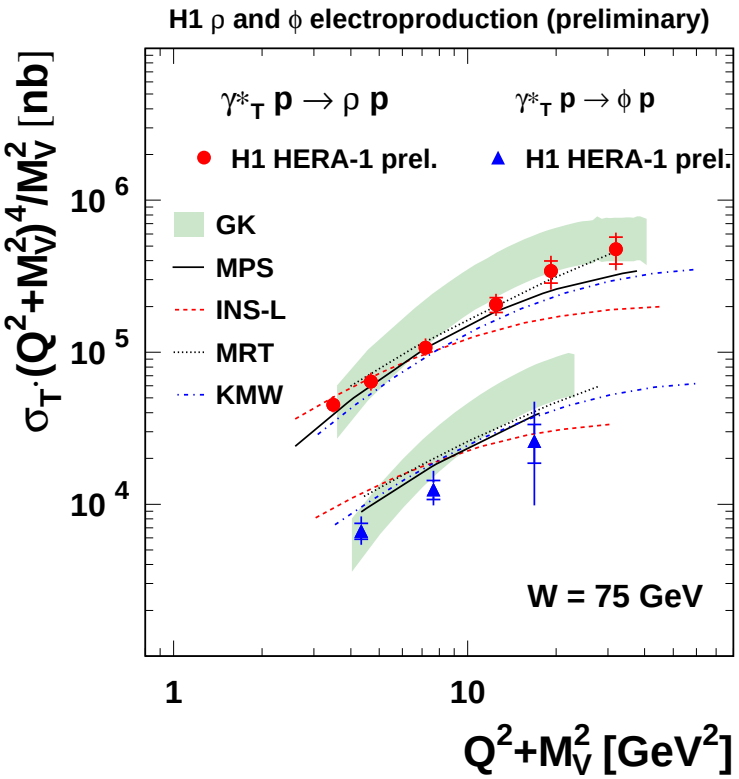
Longitudinal

$$\sigma_L \propto \frac{Q^2/M_V^2}{(Q^2+M_V^2)^4}\,[\alpha_s(\mu^2)\,G(x,\mu^2)]^2$$



Transverse

$$\sigma_T \propto \frac{1}{(Q^2+M_V^2)^4}\,[\alpha_s(\mu^2)\,G(x,\mu^2)]^2$$



- Different $Q^2 + M^2$ dependences of σ_L and σ_T ($\sigma_L = 0$ at Q^2= 0)
- Good description by models with some differences
- Effect of Q^2 dependances of $[\alpha_s(\mu^2)\,G(x,\mu^2)]^2$ visible

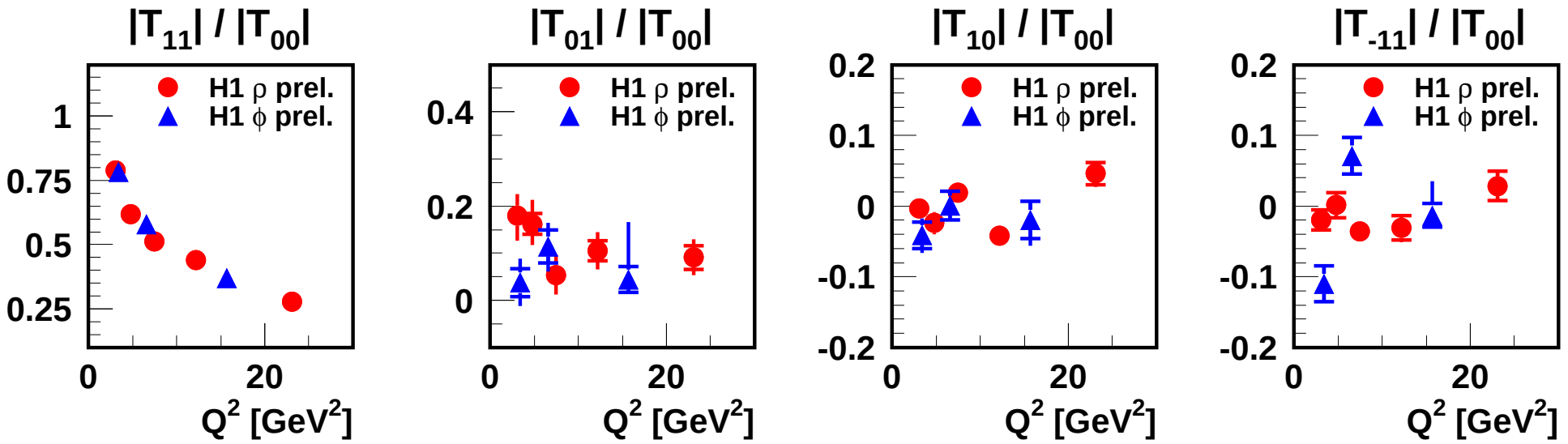
N.B.: data at fixed W → varying x with $Q^2 + M_V^2$

Polarisation - Amplitude ratios vs. Q^2

pQCD (IK):

- $T_{11}/T_{00} \propto \frac{M}{Q} \frac{1+\gamma}{\gamma}$
- $T_{10}/T_{00} \propto -\frac{M\sqrt{|t|}}{Q^2} \frac{\sqrt{2}}{\gamma}$
- $T_{01}/T_{00} \propto \frac{\sqrt{|t|}}{Q} \frac{1}{\sqrt{2}\gamma}$

γ : gluon anomalous dim.



- T_{11}/T_{00} decreases with $Q^2 \leftrightarrow \sigma_L/\sigma_T$ increases with Q^2
- $T_{01}/T_{00} > 0 \leftrightarrow$ SCHC violation
- T_{10}/T_{00} and T_{-11}/T_{00} are small

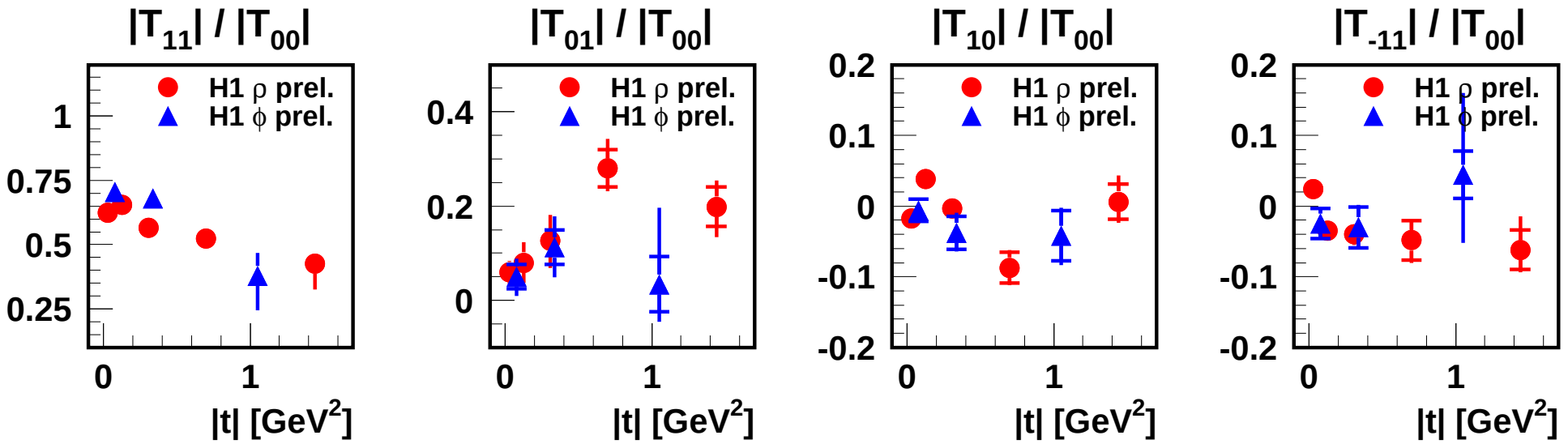
  $\Rightarrow |T_{00}| > |T_{11}| > |T_{01}| > |T_{10}|, |T_{-11}| \leftrightarrow$ hierarchy observed

Polarisation - Amplitude ratios vs. $|t|$

pQCD (IK):

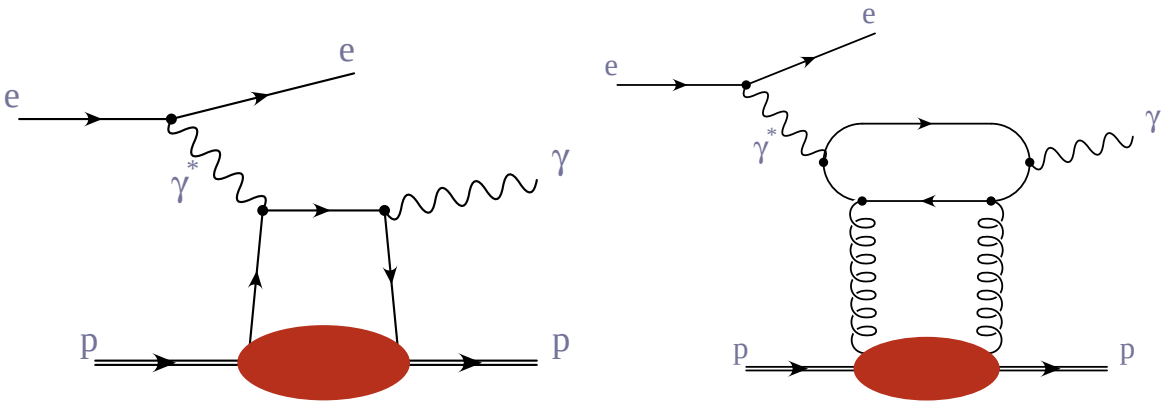
- $T_{11}/T_{00} \propto \frac{M}{Q}\,\frac{1+\gamma}{\gamma}$
- $T_{10}/T_{00} \propto -\frac{M\sqrt{|t|}}{Q^2}\,\frac{\sqrt{2}}{\gamma}$
- $T_{01}/T_{00} \propto \frac{\sqrt{|t|}}{Q}\,\frac{1}{\sqrt{2}\gamma}$

γ : gluon anomalous dim.

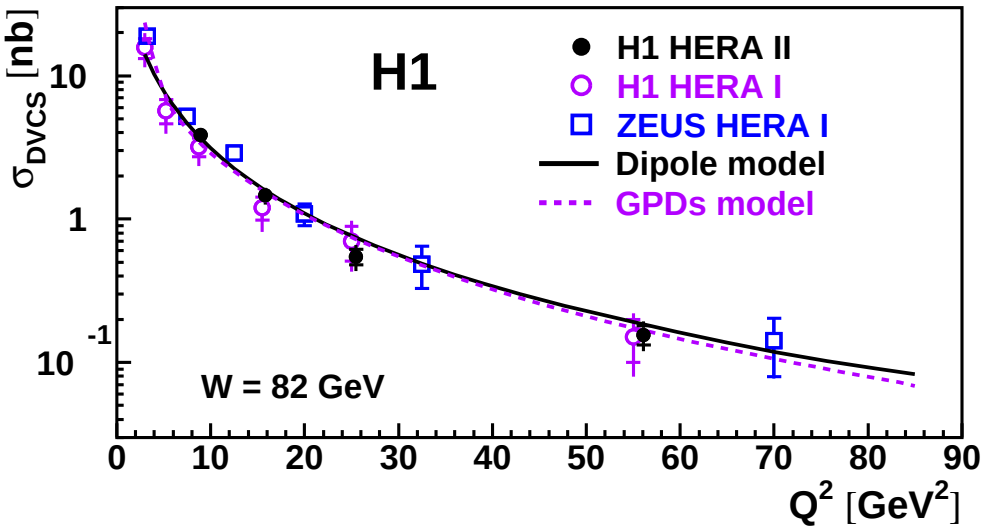


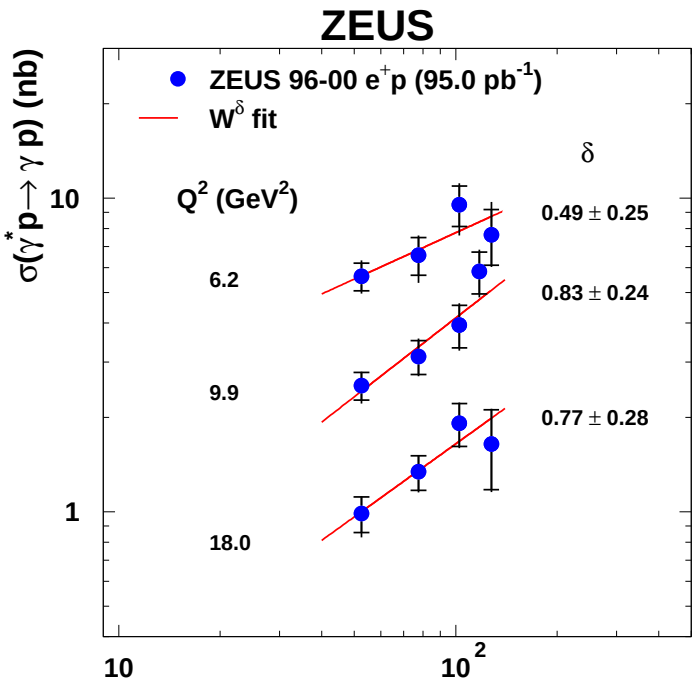
- T_{11}/T_{00} decreases with $|t|$ (cf. $b_L - b_T$)
- T_{01}/T_{00} increases with $|t| \leftrightarrow$ SCHC violation increases with $|t|$
- T_{10}/T_{00} and T_{-11}/T_{00} are small but some $|t|$dependence

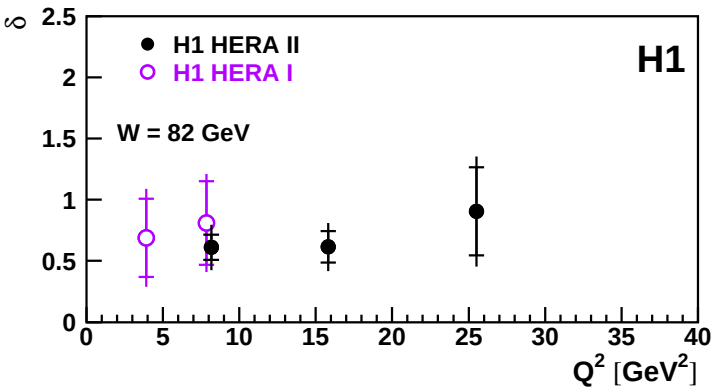
Deep Virtual Compton Scattering



- fully calculable in pQCD
- Access to the full QCD amplitude
- Constrain gluon GPDs







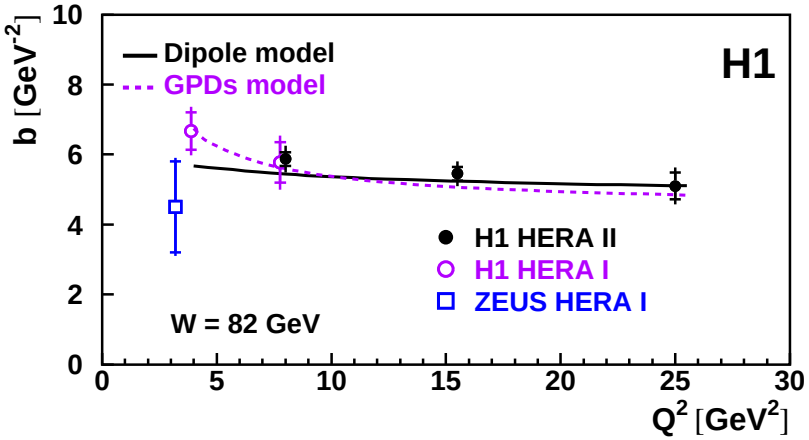
W dependence indicates a hard regime (similar to J/Ψ)

DVCS: t slope and Beam Charge Asymmetry

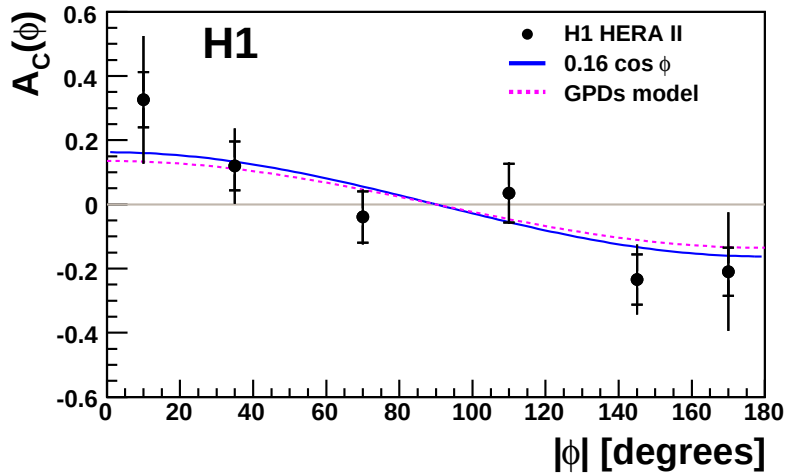
H1 measurement based on 291 pb^{-1} of HERA II data (e^+ and e^-).

- t slope as a function of Q^2

$\Rightarrow$ Similar behaviour with VM using the scale $Q^2 + M_{VM}^2$



- First DVCS BCA measured at HERA.



$$BCA \equiv \frac{\sigma(e^+p) - \sigma(e^-p)}{\sigma(e^+p) + \sigma(e^-p)} \sim p_1 cos(\Phi)$$

DVCS: QCD interpretation

- correct Q^2 dependence of the propagator and of b in the cross section:

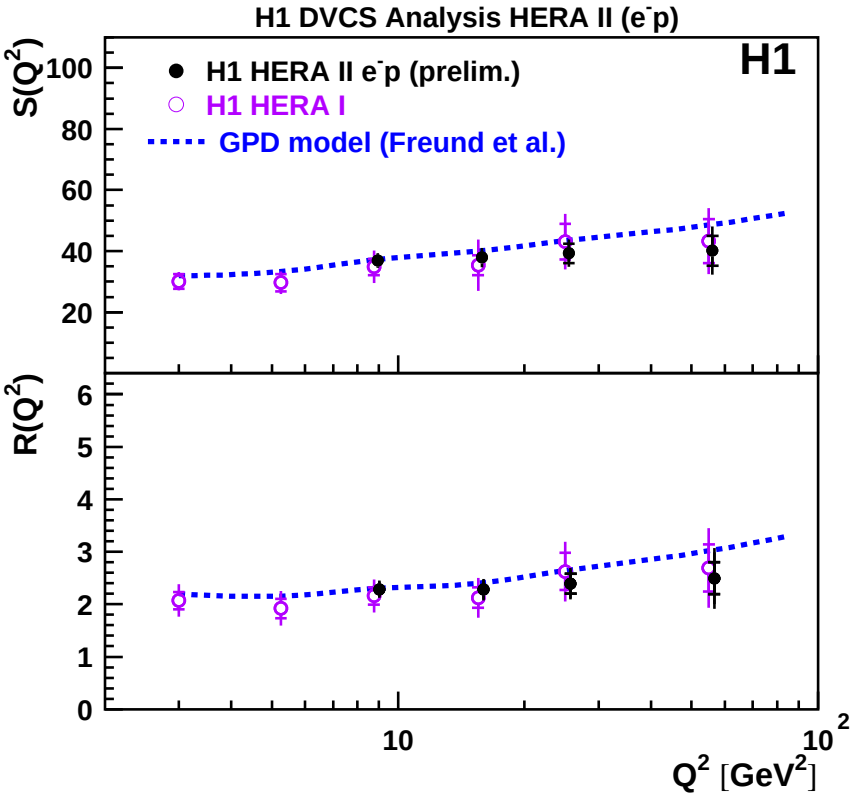
$$S = \sqrt{\frac{\sigma_{DVCS}\, Q^4\, b(Q^2)}{(1+\rho^2)}}$$

- skewing factor: around 2

$$R = \frac{\mathcal{I}m\, A(\gamma^* p \to \gamma p)}{\mathcal{I}m\, A(\gamma^* p \to \gamma^* p)} = \frac{4\sqrt{\pi\, \sigma_{DVCS}\, b(Q^2)}}{\sigma_T(\gamma^* p \to X)\sqrt{(1+\rho^2)}}$$

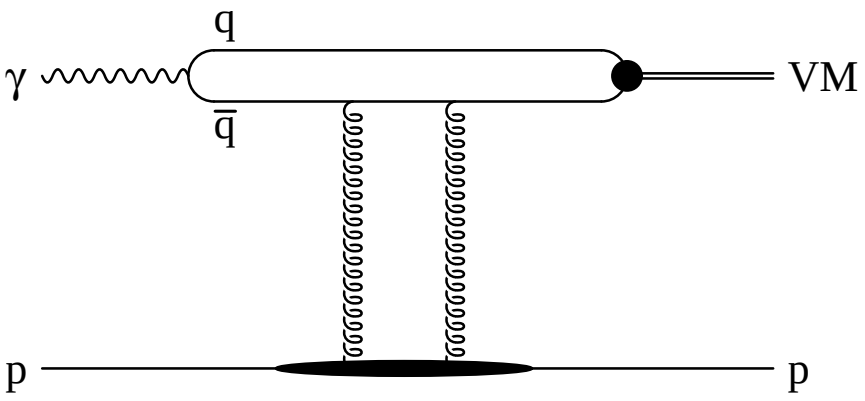
$\Rightarrow$ important skewing factor

$\Rightarrow$ Q^2 evolution close to the one of DIS (pure DGLAP)

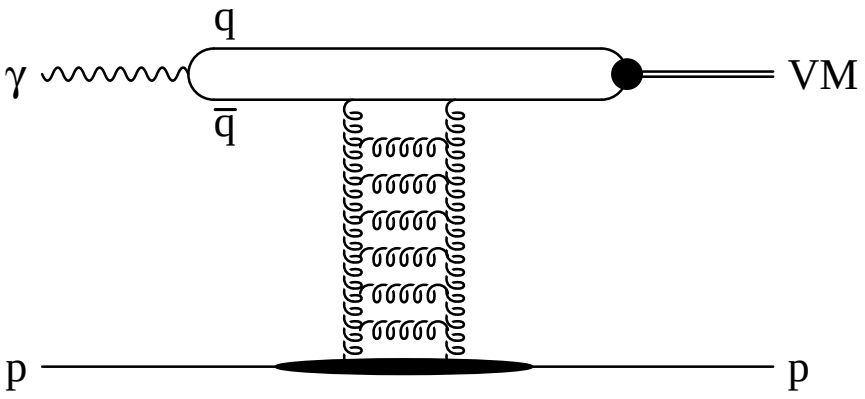


High $|t|$ Exclusive Photoproduction: Introduction

LO: 2 gluon exchange



LLA: Gluon ladder



DGLAP Evolution ($|t| < M_{VM}^2$):

Strong k_T ordering along ladder

$\rightarrow \mathrm{d}\sigma/\mathrm{dt} \sim \mathrm{e}^{\mathrm{bt}}$

$\rightarrow$ No increase of $\mathrm{d}\sigma/\mathrm{dt}$ with W

BFKL Evolution ($|t| > M_{VM}^2$):

- p_T fully transfered from p to $q\bar{q}$
- high $W \rightarrow$ small x_{Bj}

No k_T ordering in ladder

$\rightarrow \mathrm{d}\sigma/\mathrm{dt} \sim |\mathrm{t}|^{-\mathrm{n}}\,; n = 3 - 4$

$\rightarrow$ Increase of $\mathrm{d}\sigma/\mathrm{dt}$ with W

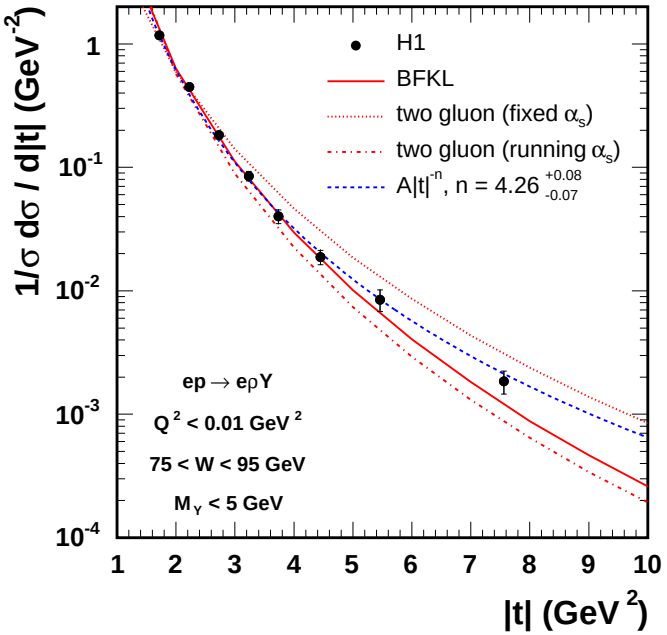
$\rightarrow$ Little shrinkage

$\rightarrow$ VM: SCHC expected

Models: Forshaw et al.

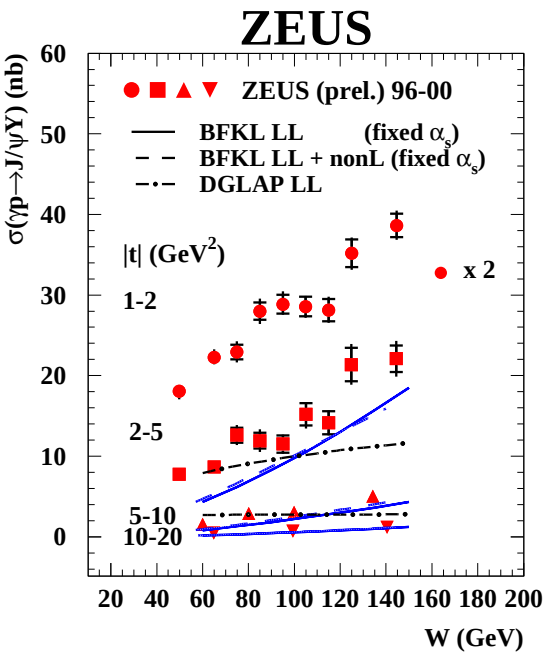
High $|t|$: Vector Meson Cross-Sections

ρ vs t



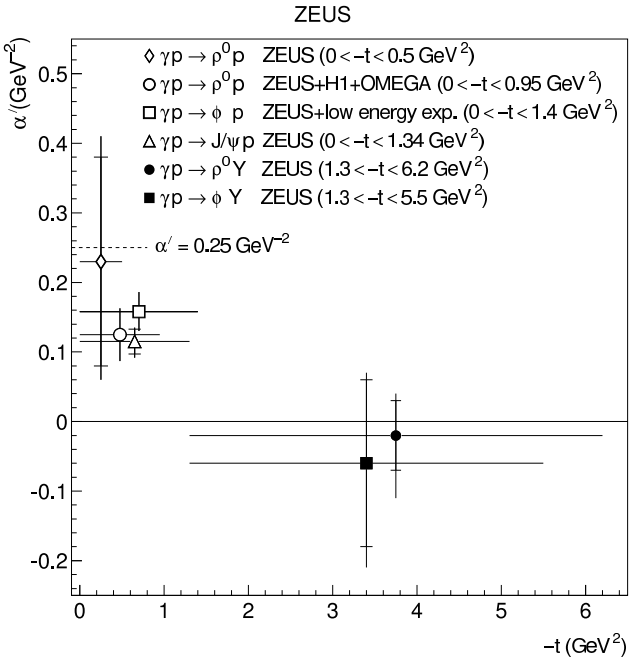
- Data follow $|t|^{-n}$
- BFKL decribes data
- 2-gluon (DGLAP-like) fails

J/ψ vs W



- Hardening of W depend. with $|t|$
- BFKL reproduces rise with W
- DGLAP fails at high $|t|$

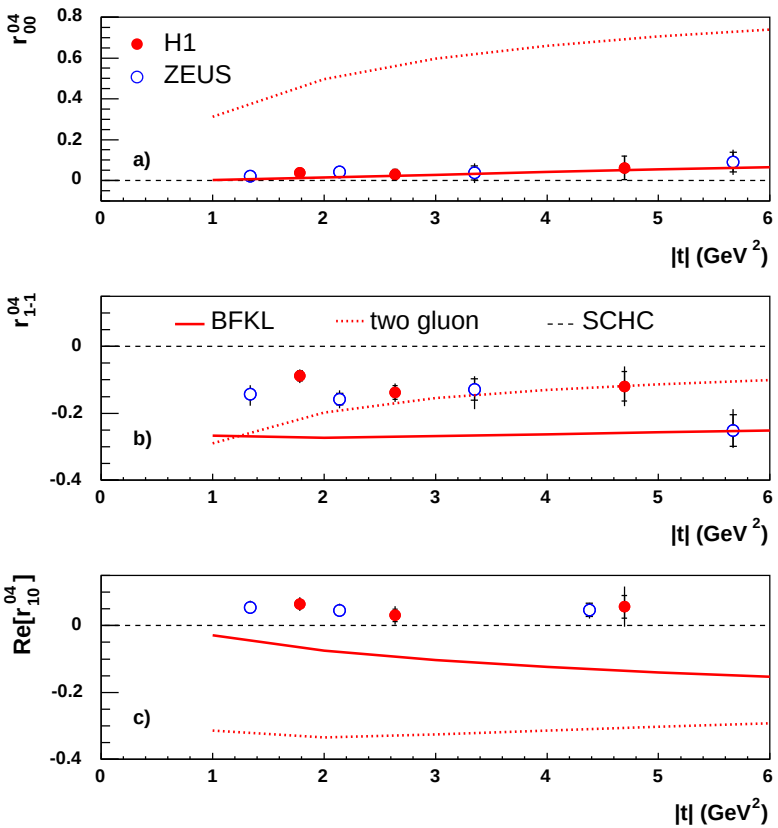
α' vs t



- α' decreases with $|t|$
- No shrinkage as expected for BFKL

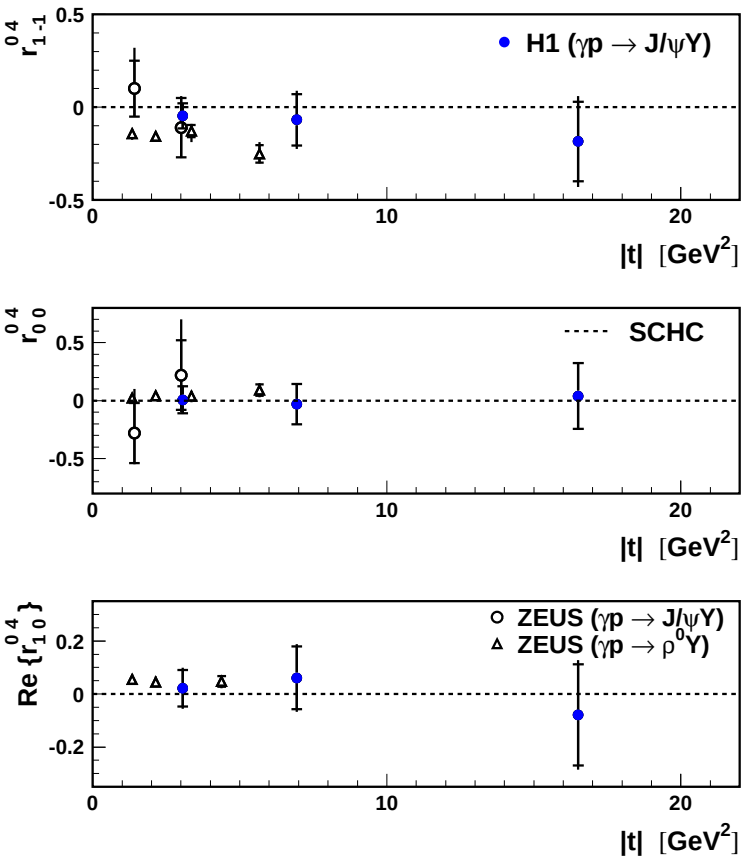
High $|t|$: Vector Meson Polarisation

ρ



- ρ data: SCHC violation for r^{04}_{1-1} and Re$[r^{04}_{10}]$
- Two-gluon and BFKL models are unable to describes data

J/ψ

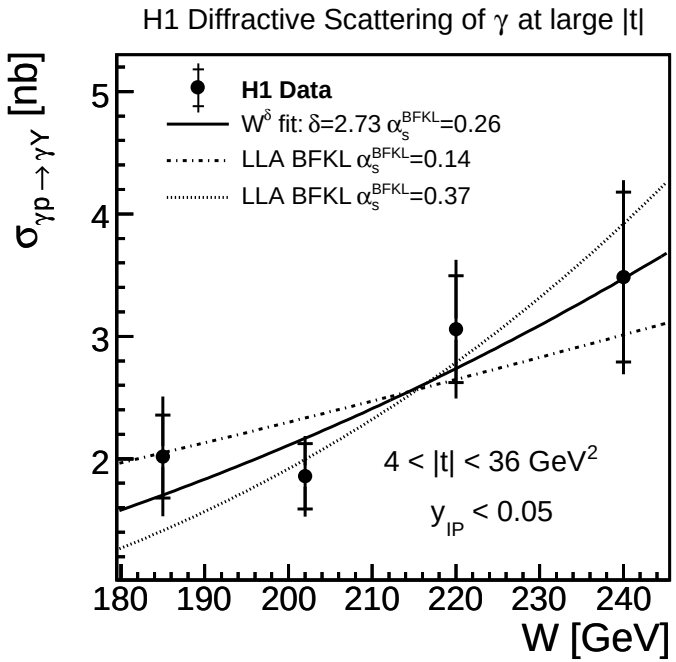


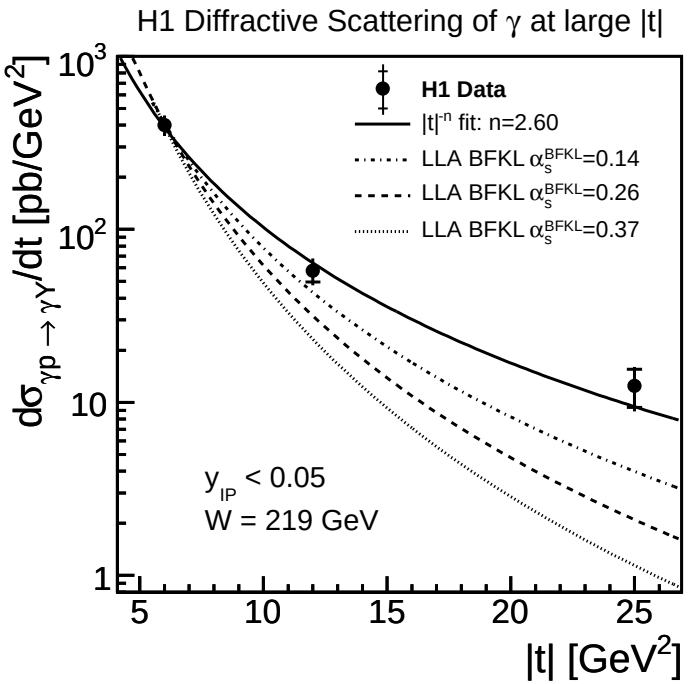
- J/ψ data: SCHC holds

$\to$ non-relativistic WF is OK

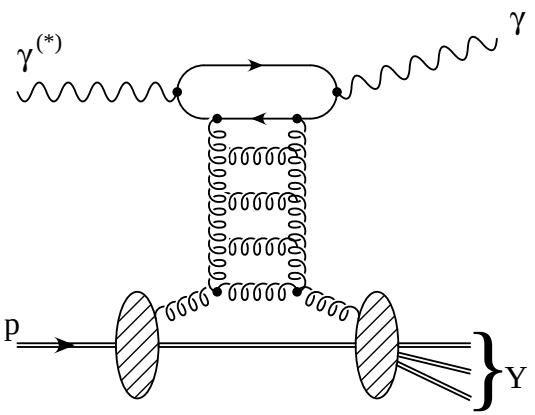
i.e. equal long. momemtum sharing between q and $\bar{q}$

High $|t|$: Diffractive High P_T Photons





First measurement of high P_T photons



H1 99-00 Data: 46 pb^{-1}

$Q^2 < 0.01\,\text{GeV}^2$
$175 < W < 247\,\text{GeV}$
$4 < |t| < 36\,\text{GeV}^2$
$y_{I\!P} \simeq e^{-\Delta\eta} < 0.05$
$E_\gamma > 8GeV$

Strong W dependence:

- W^δ fit $\rightarrow$ $\delta = 2.73 \pm 1.02^{+0.56}_{-0.78}$
  $\rightarrow$ $\alpha_s^{\text{BFKL}} = 0.26 \pm 0.10^{+0.05}_{-0.07}$ <> J/ψ : $\alpha_s \sim 0.18$
- LLA BFKL (with $\alpha_s = 0.26$) prediction describe the W dependence

t dependence:

- $|t|^{-n}$ fit $\rightarrow$ $n = 2.60 \pm 0.19^{+0.03}_{-0.08}$
- LLA BFKL too steep for t dependence

CONCLUSIONS

Important progresses in precision for VM and DVCS leading to the understanding of the underlaying dynamics

VM cross-section measurements:

- New Υ photoproduction results
- Hard regime reached only around $\mu^2 = \frac{Q^2+M^2}{4} = 5\ \mathrm{GeV}^2$ as observed in measurements of $\alpha_{I\!P}(0)$ and b-slopes.
  $\longrightarrow$ Possible soft component in σ_L up to "high" Q^2 for light VM.
- p diss. / elastic ratio: proton vertex factorisation observed

VM polarisation properties:

- Polarised cross-section and amplitude ratios have been extracted
- σ_L/σ_T increases with Q^2 and maybe with $|t|$ at high Q^2
  $\longleftrightarrow$ $|t|$ depend. expected in pQCD from $\neq$ dipole in σ_L and σ_T.
- Violation of SCHC: signicant T_{01}/T_{00} increases with $|t|$
- σ_L/σ_T decreases with ρ invariant mass
  $\longleftrightarrow$ Predicted by MRT / limited influence of VM wave function.

CONCLUSIONS

Important progresses in precision for VM and DVCS leading to the understanding of the underlaying dynamics

DVCS:

- W dependence indicates hard regime
- Significant skewing factor measured $\longrightarrow$ GPD
- First Beam Charge Assymetry measured

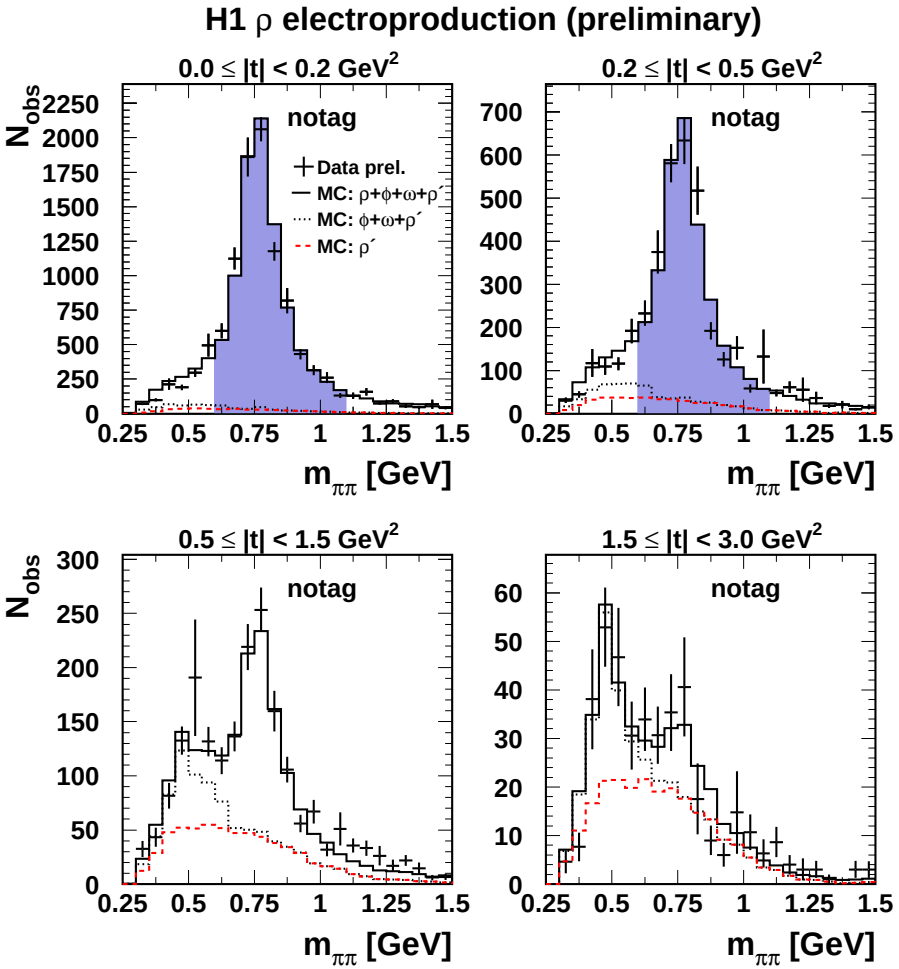
High $|t|$:

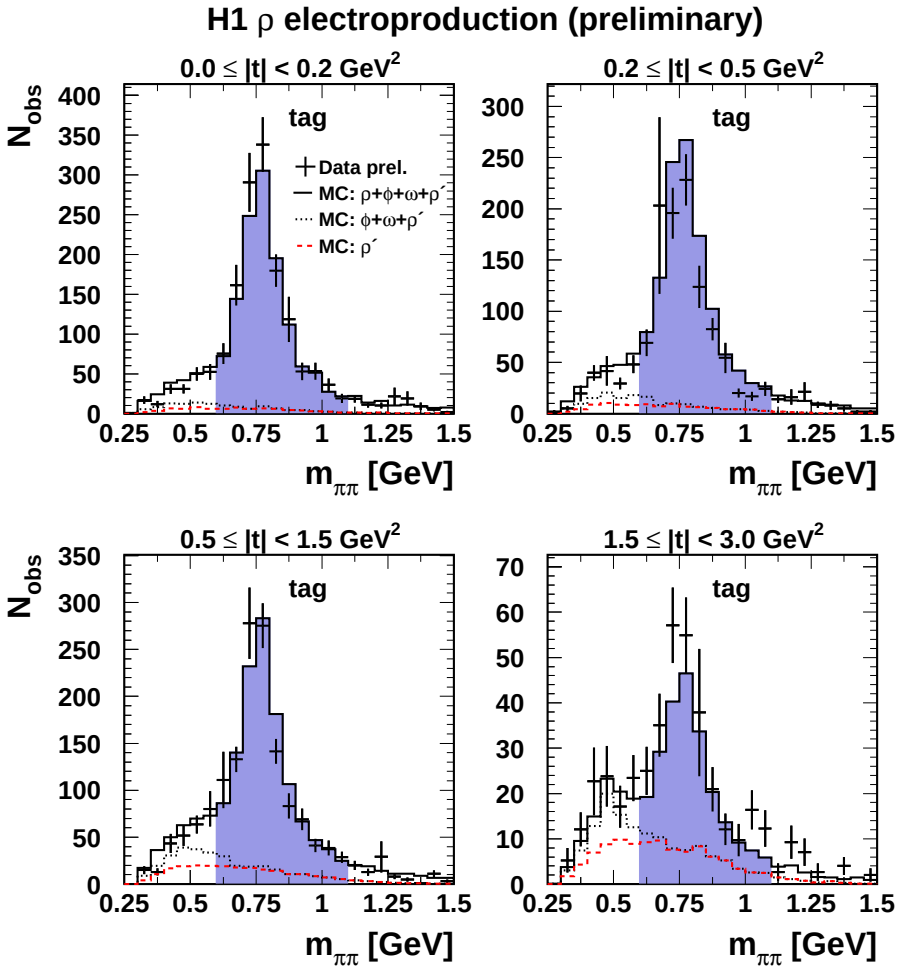
- High $|t|$ VM data are showing BFKL like behaviours
- High P_T photons measurement roughly agree with BFKL

pQCD models:

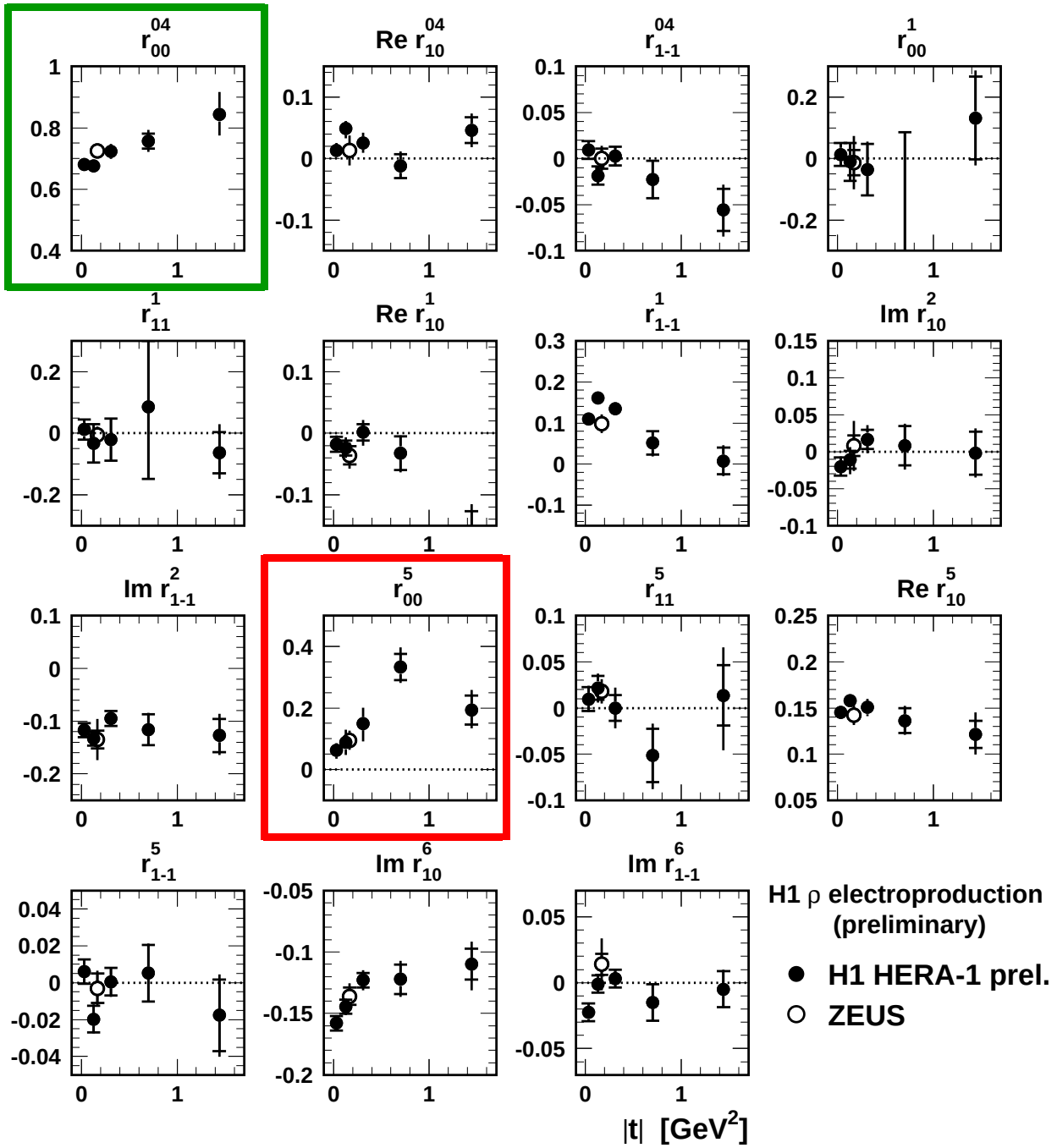
- GPD and dipole models describe main features of data
- Some differences in details

H1 background substraction





ρ Polarisation - SDMEs vs. $|t|$



- r^{5}_{00} increases with $|t|$
  $\leftrightarrow$ SCHC violation increases with $|t|$

- r^{04}_{00} increases with $|t|$
  $\leftrightarrow$ similar effects for r^{1}_{1-1}, Im r^{2}_{1-1}, Re r^{5}_{10} and Im r^{6}_{10} (in SCHC)

Polarisation - Retrieving Amplitude ratios

Assume purely imaginary amplitudes $\longrightarrow$ phase = $\pm$ 1 !

$\longrightarrow$ Extract $|T_{11}|/|T_{00}|$, $|T_{01}|/|T_{00}|$, $|T_{10}|/|T_{00}|$ and $|T_{-11}|/|T_{00}|$ from fit to the 15 SDMEs:

$$
\begin{aligned}
r^{04}_{00} &= B\,(\varepsilon + \beta^2) \\
\mathrm{Re}\, r^{04}_{10} &= B/2\,(2\varepsilon\delta + \beta\alpha - \beta\eta) \\
r^{04}_{1-1} &= B\,(\alpha\eta - \varepsilon\delta^2) \\
r^{1}_{00} &= -B\,\beta^2 \\
r^{1}_{11} &= B\,\alpha\eta \\
\mathrm{Re}\, r^{1}_{10} &= B/2\,\beta(\eta - \alpha) \\
r^{1}_{1-1} &= B/2\,(\alpha^2 + \eta^2) \\
\mathrm{Im}\, r^{2}_{10} &= B/2\,\beta(\alpha + \eta) \\
\mathrm{Im}\, r^{2}_{1-1} &= B/2\,(\eta^2 - \alpha^2) \\
r^{5}_{00} &= \sqrt{2}B\,\beta \\
r^{5}_{11} &= B/\sqrt{2}\,\delta(\alpha - \eta) \\
\mathrm{Re}\, r^{5}_{10} &= B/(2\sqrt{2})\,(2\beta\delta + \alpha - \eta) \\
r^{5}_{1-1} &= B/\sqrt{2}\,\delta(\eta - \alpha) \\
\mathrm{Im}\, r^{6}_{10} &= -B/(2\sqrt{2})\,(\alpha + \eta) \\
\mathrm{Im}\, r^{6}_{1-1} &= B/\sqrt{2}\,\delta(\alpha + \eta)
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= |T_{11}|/|T_{00}| \\
\beta &= |T_{01}|/|T_{00}| \\
\delta &= |T_{10}|/|T_{00}| \\
\eta &= |T_{-11}|/|T_{00}|
\end{aligned}
$$

$$
\begin{aligned}
B &= \tfrac{1}{N_T + \varepsilon N_L} = \tfrac{R}{1+\varepsilon R} \\
N_T &= \alpha^2 + \beta^2 + \eta^2 \\
N_L &= 1 + 2\delta^2
\end{aligned}
$$